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THESIS

AN ANALYTIC FRAMEWORK FOR THE WAR OF IDEAS

by

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September 2006

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We develop models for the spread of two opposing ideologies in a closed population based on epidemic models. Based on different interaction rules, we study deterministic and stochastic models of the problem. The goal of our work is to provide a tractable analytical framework for each situation, and to analyze the effect of different initial conditions on the proportion of the population affiliated with each ideology after a large time interval.

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AN ANALYTIC FRAMEWORK FOR THE WAR OF IDEAS

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ABSTRACT

We develop several models for the spread of two opposing ideologies in a closed population based on epidemic models. Based on different interaction rules for the mechanism of how the ideologies are spread, we study the results of deterministic and stochastic models. The goal of our work is to provide a tractable analytical framework for each hypothesized set of social interactions, and to analyze the effect of different initial conditions on the proportion of the population affiliated with each ideology after a large time interval. The models developed herein are designed to give decision makers an insight into a complex process.

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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

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EXECUTIVE SUMMARY

One of the objectives listed in the 2003 National Strategy for Combating Terrorism is to win the "War of Ideas".[1] Our work seeks to place an analytic framework around this war. Our goal is to create a methodology for considering alternatives and some concrete metrics with which to compare courses of action. Our fundamental assumption is that one-to-one (i.e. interpersonal) communication is the most important in spreading ideas.

Our tools are deterministic and stochastic models originally developed for infectious diseases and rumor propagation. The fundamental idea is that when two people in the population connect (either through direct contact or phone, email, etc), the ideology may be spread. These models are similar to traditional epidemic models. Many extensions to the idea of the spread of ideology as disease are possible and some are explored in our work.

We extend previous work by placing ideology in a greater social context. We introduce two, diametrically opposed ideas and model their flow. We refer to the proponents of these ideas as the *supporters* and *contrarians*. We consider the case where both the supporters and contrarians openly vie for a greater share of support from the public. This case is similar to a political campaign in the United States without the influence of media. We also consider the case where the supporters are able to openly propagate their message, but the contrarians only interact when supporters try to convert them. We believe this is the case where there is a small but dedicated extremist sub-population. We find that under the model assumptions a relatively small number of contrarians are required to overcome a large increase in the supporters

[1] National Strategy for Combating Terrorism, 23.

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I. INTRODUCTION

> From the beginning, the War on Terror has been both a battle of arms and a battle of ideas – a fight against the terrorists and against their (ideology). In the short run, the fight involves using military force and other instruments of national power… In the long run, winning the war on terror means winning the battle of ideas, for it is ideas that can turn the disenchanted into terrorists.
>
> -- **National Security Strategy of the United States, March 2006**

> It is ideological belief, reinforced by propaganda operations, that convinces recruits and supporters that their actions are morally justifiable. All the instruments of national power play a role in undermining ideological support… Targeting ideology includes amplifying the voices of those who promote alternative ideas
>
> -- **National Military Strategic Plan for the War on Terrorism, February 2006**

One of the objectives listed in the National Strategy for Combating Terrorism (February 2003) is to "Win the War of Ideas". Specifically:

> Together with the international community, we will wage a war of ideas to make clear that all acts of terrorism are illegitimate, to ensure that the conditions and ideologies that promote terrorism do not find fertile ground… and to kindle the hopes and aspirations of freedom of those in societies ruled by the sponsors of global terrorism.

Clearly, the United States, at the highest level, values winning the War of Ideas, and for good reason. Winning the war of ideas is possibly the single most important action the US can take to further the National Interest. Winning the War of Ideas would have the immediate benefit of significantly reducing the requirement for military force. Winning the War of Ideas will cost far less than imposing our will by force.

To our knowledge, little work has been done by Operations Research practitioners on this important problem. At first glance, the statement of this problem lends itself more readily to the social scientists. Operations Research practitioners do in fact have significant tools to contribute. The nature of the current war is new and different than past conflicts and requires all of us to step outside our traditional boundaries.

Our examination of the war of ideas takes place at the micro level, focusing exclusively on one-to-one communications. These may be instantiated either in person or via a communication channel such as email or telephone. We do not consider the effect of mass media. Apart from mass media, we believe that the most important communication mode in swaying a person's ideology is interpersonal communications. We use a simple model to describe a person's ideology; they can either be neutral, for, or against. The object of their ideology is immaterial for our purposes. We view the spread of ideology to be an epidemic with basically two mutually exclusive populations: supporters and contrarians. We extend classic models of epidemiology and rumor propagation to suit this new purpose.

Our goals in this thesis are threefold. First, we seek to provide an analytic, descriptive framework for decision making in the War of Ideas. To clarify, we will highlight what our model is not: It is neither a proscriptive set of equations to be used in a vacuum, nor a predictive model that projects population behavior. Rather, it is a descriptive tool to help understand cause-and-effect relations in the context of 'war of ideas'. Secondly, we provide coarse-detail, deterministic and stochastic models for the propagation of ideology, which may be used as a base case for validating complex simulations. The US Army is currently in the process of examining proposals for simulation models to study the War of Ideas[2]. Finally, our model is mathematically interesting because it introduces competition in the context of rumor propagation. In addition to being interesting from a purely mathematical point of view, the models developed herein represent a step forward in the study of social interactions as stochastic processes.

Chapter II provides a short review of current models for the spread of rumor/ideology. Chapter III develops our models, called *NSRL* and *NSRCL* and explores their behavior deterministically. Chapter IV explores the *NSRL* model probabilistically using both a Markov chain as well as a diffusion approximation. Chapter V presents our conclusion and recommends future research.

[2] U.S. Army Small Business Innovative Research initiative A06-094 http://www.acq.osd.mil/osbp/sbir/solicitations/sbir062/army062.htm

II. LITERATURE REVIEW

> The mathematical theory of epidemics appears at present to be developing fairly rapidly. The theory is only likely to have valuable applications in so far as it is developed in the context of a proper understanding of the epidemiological realities. Many of the models… are of necessity oversimplified: nevertheless, useful results are readily available.
>
> – Norman T. J. Bailey, 1975.

A. RUMORS AS A SUBSET OF INFECTIOUS DISEASE MODELS

The analytic models of rumor (used interchangeably with *information)* propagation are closely linked to the analytic models of infectious diseases. Indeed, one of the earliest works in this field was motivated by the desire to control 'psychological infections' and was developed in the 1920's (Daley and Gani, 1999). The similarity between rumor propagation models and models for infectious disease is that they both describe the behaviors of actors; we use the term 'actor(s)' in our development to mean the atomic level of social interaction. Actors may be thought of as individuals, but may also be thought of as family units or even entire villages. This definition of an actor is encouraged both in Maki and Thompson (1973) as well as by Wasserman and Faust (1994). The development of models for infectious disease focuses on these actors interacting with each other in accordance with some well-defined (deterministic or stochastic) set of rules. In this context it is useful to think of spreading a rumor as 'infecting with knowledge'. We review two models for the spread of rumors: The Maki-Thompson [MT] (Maki and Thompson (1973)) and Daley-Kendall [DK] (Daley and Kendall (1964)) models.

B. MODELS FOR INFECTIOUS DISEASE: THE BASIC FRAMEWORK

Since our development depends heavily on the existing mechanics of infectious disease models, it is useful to describe how infectious disease models work. Particularly useful subsets of infectious disease models are those which partition the population into three classes. The following table explains the so-called "S-I-R" (after Waltman, 1970) models. A rigorous development of the basic mathematical models for infectious diseases is contained in Bailey (1975).

Population	Definition
S	The ***Susceptible*** population. These are individuals who are not yet infected, but may be at some future time. In rumor models they are called the ignorant population.
I	The ***Infective*** population. These individuals are the active carriers and spreaders of disease. In rumor models they are called spreaders.
R	The ***Removed*** population. These individuals have been infected and have since been removed (either by death or immunity). In rumor models they are called stiflers.

Table 1: Definitions of States

A characteristic of S-I-R models is a finite population of size N + 1, representing N initial susceptibles and one initial carrier. Furthermore, this population exists within a closed system. Because the future state of the system is dependant only upon the current state, this model exhibits the Markov property.

This type of model belongs to the class of *strictly evolutionary* processes because of the following property: once a state, as defined by the vector $X_t = \left[S_t, I_t, R_t\right]$, is visited it may never be visited again. In both the [DK] and [MT] models, which are discussed in the next section, the members of the population who make the transition from **S** to **I** eventually reach the absorbing state of **R**. For both deterministic and stochastic models, at the end of the rumor spread, the population is partitioned into ignorants (**S**) and stiflers (**R**), with no spreaders (**I**) left. In the stochastic version of this model, an embedded Martingale is present as noted by Daley and Gani, (1999). We use the embedded Martingale in Chapter IV to study the asymptotic distribution of the process without explicitly specifying the associated Markov chain.

Belen and Pearce (2004) note that formally, the models described by the two models ([DK] and [MT]) are the same.

C. RUMOR PROPAGATION

1. The Basic Model

The basic mechanics of rumor propagation are described in (Daley and Kendall, 1965) and are as follows: The population is partitioned into three mutually exclusive and collectively exhaustive classes as in Table 1. The classes within the population have the following interpretation:

S: ***Susceptibles*** (interchangeably, the ***ignorants***). Those who do not know the information but may in the future.

I: The ***active spreaders***. Those who know the information and are actively disseminating it to those individuals they meet

R: The ***stiflers***. The individuals who know the information and are no longer spreading it.

The same definitions are used in the [MT] model. A stochastic version of this same model is provided as Appendix B. The properties of the [DK] model have been explored by several authors, including Watson (1987), Pittel (1990), and Hayes (2005).

The following table describes the transitions between actor A and actor B:

	ACTOR "A"			
ACTOR "B"		S	I	R
	S	No Change	B -> I A: no change	No Change
	I	A -> I	A-> R B-> R	B-> R
	R	No Change	A -> R	No Change

Table 2: Daley-Kendall Transition rules (after Hayes, 2005)

The [MT] model is very similar, except that when A and B are both members of class **I**, only one becomes a stifler as a result of the interaction.

Equations (1) present the deterministic Daley-Kendall model (Daley and Gani, 1999):

$$
\begin{aligned}
S' &= -SI \\
I' &= SI - I^2 - IR \qquad (1) \\
R' &= I^2 + IR
\end{aligned}
$$

The numerical solution of equations (1 is presented as Figure (1) with a population size of 1000, and 1 initial spreader.

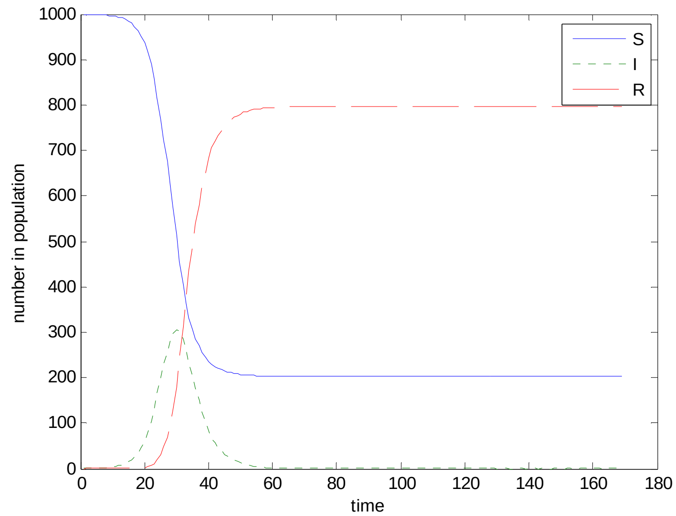

Figure 1. Daley-Kendall Model

2. Stifling Mechanisms

Common experience tells us that rumors do not generally spread indefinitely. To force the rumor to end at some time, **I** must be a transient state. The mechanism by which actors leave the class **I** is called *stifling*.

a. Models Which Lack Stifling Mechanisms

The simplest models lack stifling. An example of this type of model is attributed to Bailey (1957) and is described in chapter 9.2 of Bartholomew (1967). This model can be derived directly from [DK] by neglecting the class ***R***. In a deterministic model which lacks stifling, the number of spreaders increases as a logistic *S*-curve until only one ignorant remains[3]. At this stage the rumor is considered to have 'fully propagated'. In the stochastic model, the rumor is fully propagated when no ignorants remain.

3 In the deterministic model, the number of ignorants asymptotically reaches zero. Therefore, a limiting argument on one ignorant remaining must be used.

In the no-stifling formulation, the population consists only of $\boldsymbol{S}$ and $\boldsymbol{I}$ types, and because we assume a known fixed population N, this model collapses to a one dimensional case because $\boldsymbol{S}(t) = N - \mathbf{I}(\mathrm{t})$. This finite population property is frequently exploited in the subsequent Chapters of this Thesis to simplify calculations. Equations (2) and (3) contain the equation for the deterministic simple epidemic in continuous time (Daley and Gani, 1999) and its solution, respectively. Figure (2) presents a graph of equation (3)

$$\frac{dI}{dt} = \beta SI = \beta I(N - I) \tag{2}$$

$$I(t) = \frac{I_0 N}{I_0 + (N - I_0)e^{-\beta N t}} \tag{3}$$

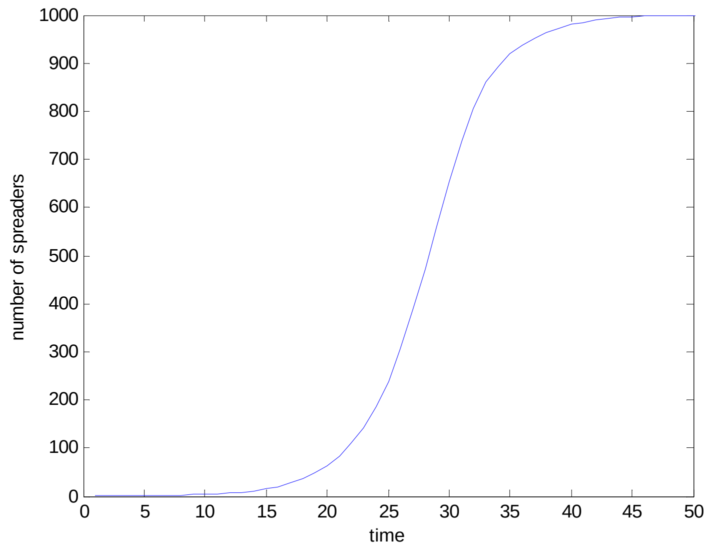

Figure 2. Deterministic non-stifling sample path

b. Daley-Kendall Stifling

Everyday experience tells us that not all news and rumors will propagate forever. A *stifling mechanism* is necessary if the rumor is to spread for a finite period of time (Daley and Gani, (1999)).

In Daley and Kendall's model, stifling occurs when a spreader meets another spreader, or when a spreader meets a stifler. In the instance of the [DK] model,

when two spreaders meet, both become stiflers. In the instance of the [MT] model, both only one spreader becomes a stifler.

D. THEORETICAL RESULTS

An interesting result was first demonstrated in the deterministic version of the [DK] model, later extended to the stochastic version and finally (Belen and Pearce 2004) generalized to the [MT] model: Given one initial spreader, the proportion of ignorants at the end of the rumor propagation converges to approximately 20%. This result is published in Daley and Kendall (1965), and is confirmed in Pittel (1990) as well as Hayes (2005) and is mentioned by Zanette (2005). A proof is contained in the aforementioned papers, as well as in Daley and Gani (1999). Additionally, as shown in Daley and Gani (1999), for ‘small’ populations, there is a probability of very limited spread due to the spreaders encountering other spreaders immediately at the start of rumor propagation. As the population tends towards infinity, the probability of this small spread phenomenon tends towards zero (Gani and Daley, 1999).

The small spread phenomenon is of great interest when studying the stochastic behavior of these processes. We will discuss this issue extensively in Chapter IV.

E. POPULATION DYNAMICS

The epidemic-style models of rumor propagation assumes a homogeneous mixing of the population and results in diffusion according to the Law of Mass Action (Daley and Gani, 1999). Other population mixing dynamics are possible and have been explored by several authors, c. f. Kress (2005). In the absence of any information about the interaction between the actors, homogeneous mixing is an appropriate assumption. With limited information about the sociodynamics of the population, mechanisms like those proposed by Kress (2005) are appropriate.

If more detailed information about the underlying social organization exists, then a different approach is warranted. Zanette (2005) explores the propagation of rumors on both ‘small world networks’ (deterministic) as well as ‘dynamic small worlds’ (stochastic). Zanette’s approach builds both on the [DK] model.

F. DETERMINISTIC VS. STOCHASTIC MODELS

The model of rumor propagation may be explored by either deterministic models or stochastic models. The approach taken depends on the level of analysis the researcher seeks to conduct and also depends to some extent on the complexity of the problem. A deterministic modeling approach may be taken either in discrete or continuous time. The discrete time model is implemented as a set of difference equations, describing system behavior at time 0, Δt, $2\Delta t, \ldots$. The limit of the discrete time model as Δt approaches zero is a continuous time model. The continuous time model is implemented as a set of differential equations. With either the discrete or continuous time deterministic models, there will be only one sample path for each set of initial conditions and parameter values. Given the state vector of the number of individuals in the population in each state (S, I, and R) at any time, the future state vector of the number of individuals in each state is completely specified.

Stochastic models take a different approach. The stochastic process in discrete time, modeled by a discrete-time Markov chain focuses on the probabilities of transition between states. In sharp contrast to the deterministic model, knowing the state vector of the number of individuals in each state (S, I, R) at any time does not (unless the state described is absorbing) specify a single future state of the system. Many sample paths are possible. The stochastic approach will yield conditional probabilities of future states given the current state. As we are only studying the asymptotic behavior of the process as $t \uparrow \infty$, we do not have to make any specific statement about the distribution of transition times.

Although generally more difficult to implement and analyze, stochastic models provide a richness that deterministic models lack. Stochastic models give us not only the expected outcome of the rumor process, but the distribution of outcomes as well. Deterministic models have the advantage of being easy to formulate and implement. They are also the only choice in situations where the state space that arises in the stochastic formulation is simply intractable. The following quote from Bailey (1975) shapes, in our opinion, the trade-offs between purely deterministic and stochastic models:

> The mathematician's first choice was always to go for stochastic modeling, since it was more interesting; and in any case there were plenty of examples, such as the analysis of family data, for which deterministic models would be virtually meaningless. [However] … a somewhat more balanced judgment is possible. When numbers of susceptibles and infectives are both large and mixing is reasonably homogenous, a deterministic treatment is likely to be fairly satisfactory as a first approximation, though even here it may be difficult to account for large-scale phenomena…[4]

[4] Bailey, . The Mathematical Theory of Infectious Diseases , 25.Bailey, Norman T.J.

III. DETERMINISTIC MODELS

A. MOTIVATION, PRELIMINARIES AND NOTATION

Deterministic models in the form of systems of differential equations are of great importance in the study of evolutionary processes such as ideological propagation and epidemics. Several texts, notably Anderson and May (1992), are devoted solely to their development and implementation. A deterministic model is one where, at each infinitesimal time step, the underlying (probabilistic) process is assumed to take on its expected value, resulting in a single path through the state space.

This class of models is attractive for several reasons. First, they are easy to implement. If the given system does not yield a closed-form solution, we may still investigate its asymptotic (long range) behavior or solve the system numerically using one of many available methods. In general, numerical solution is very fast.

Because deterministic models do not respect the integer nature of populations as stochastic models do, their qualitative behavior is invariant to the population size. This feature makes them unsatisfactory for small populations where random fluctuations may be significant and therefore stochastic models should be used. However, in Chapter IV we reference a theoretical result (Barbour, 1972), which shows that under certain assumptions for large populations, the stochastic mean converges to the deterministic solution.

The models we develop in the following section will be used in this Chapter and Chapter IV. In these models we identify several population classes that interact. For any population class X we denote the initial population size as X_0, and the long term limiting size (if such limit exists) as X_∞. We use $X = X(t)$ to denote also the size of the population at time t. In general we consider four population classes:

- ***N***: *Neutrals* who hold no firm ideology or opinion and who may be susceptible to ideological influence;
- ***S***: *Active Supporters* who actively spread pro-government ideas;

- ***R***: *Latent Supporters* who support the government but do not actively spread their ideas;
- ***C***: *Active Contrarians* who actively spread anti-government ideas;
- ***L***: *Latent Contrarians* who oppose the government but do not actively spread their ideas;
- ***P****=N+S+R++C+L*: *Total Population.*

In Section B we develop two models called NSRL and NSRCL. In Section C we examine some special cases of these models which possess closed-form solutions. In Section D we present numerical results.

B. THE *NSRL* AND *NSRCL* MODELS

We develop two models which capture the dynamics of the spread of ideas. Although the two models represent different situations, they share a similar structure. Both models are posed here in a deterministic form for closed populations; that is, individuals do not enter or leave the population. This assumption is realistic in remote isolated communities. A generalized case, allowing for migration in and out of the population is a natural and feasible extension, but it is one we do not pursue here.

1. The *NSRL* Model

The *NSRL* model considers the following situation: There is a message, which is overtly being passed by the active supporters (***S***). There are a certain number of latent contrarians (***L***) in the population who are antagonistic to the supporters, but who do not actively spread their views. These contrarians are passive until they meet an active supporter, in which case they will attempt to convert him to their side. The implication is that individuals with a counter-attitude do not want to advertise, but when an active supporter attempts to recruit a contrarian, the counter-message by the contrarian overwhelms (with some probability) and converts the supporter to the contrarian way of thinking. The *NSRL* model is asymmetrical in the sense that the supporters and contrarians sides behave differently (e.g., there are no overt contrarians) as opposed to a

symmetric model, in which both sides have overt members. We have the situation of an insurgency or popular front in mind in the development of the *NSRL* model.

Initially the majority of the population is in class ***N*** (neutrals) with some proportion belonging to class ***L*** and some proportion belonging to class ***S***.

We assume that homogeneous mixing takes place in the population, meaning that the probability of interaction between two members is dependant on the proportions and attributes of the populations only, and not other factors such as time of day, location, etc. An extension involving time variant mixing parameters is feasible, but not pursued here.

An *interaction* occurs when one member (of any class) meets another member (also of any class), and the result of this meeting is that at least one of the two members may change its class. Table 3 presents the seven possible interactions in the *NSRL* model. The bold letters in the table are the row and column indices (classes). The entry corresponding to the row/column pair describes the possible class change that may occur. For example, 'N->S' means that a member of the ***N*** population class may leave its class and join the ***S*** class. We will use this notation for the remainder of the chapter.

	N	***S***	***R***	***L***
N	--	*N->S*	--	--
S	*N->S*	*S->R* *S->R*	*S->R*	*S->L*
R	--	*S->R*	--	--
L	--	*S->L*	--	--

Table 3: NSRL Interactions

Figure 3 is a pictorial representation of Table 3.

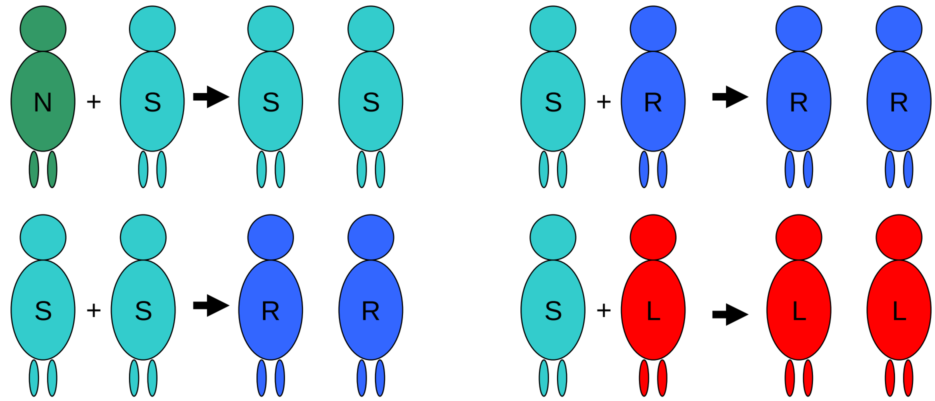

Figure 3. NSRL Interactions

It is evident that class **S** interacts with every other class (including itself). All other classes only interact with class **S**. Table 3 leads directly to the following system of Ordinary Differential Equations (ODEs):

$$\frac{dN}{dt} = -\beta_1 NS$$
$$\frac{dS}{dt} = \beta_1 NS - \beta_2 S^2 - \beta_3 SR - \beta_4 SL$$
$$\frac{dR}{dt} = \beta_2 S^2 + \beta_3 SR$$
$$\frac{dL}{dt} = \beta_4 SL$$

with the initial conditions: (4)

$$N_0 = (1 - (r_S + r_L))P$$
$$S_0 = r_S N$$
$$R_0 = 0$$
$$L_0 = r_L N$$
$$r_S + r_L \leq 1$$

where the β_i are parameters describing the intensity of each class' interaction with the other classes, N_0, S_0, R_0, L_0 are the initial sizes of each class, and r_S, r_L are the proportion of the population that is initially in classes ***S*** and ***L***, respectively.

Note that it is possible to fix the parameter *P* and appropriately adjust the β_is to achieve the same result. We will employ this technique and use the 'normalized' system with *P*=1 in the analysis section.

We allow for the possibility that members of different populations may act with different intensities. For example, the active supporters (***S***) may aggressively seek out Neutrals (***N***), but avoid Retired (***R***) supporters. Note that if $r_L = 0$, the model posed here collapses to that proposed by Daley and Kendall (1964). If $r_S = 0$, then the initial state is an absorbing state and no change takes place.

2 The *NSRCL* Model

This model has a similar development to the *NSRC*, with one important difference: it is symmetrical. We add the class '*C*' of active contrarians and now the two sides have symmetric rules.

The situation described by this model differs from the asymmetrical *NSRL* model in that both messages are spread openly. This situation may be one of political change, such as the formation of a government or an election. The active supporters and active contrarians may be thought of as belonging to different political parties, voting for different candidates, etc. The interactions in this model are described in Table 4.

	N	***S***	***R***	***C***	***L***
N	--	*N->S*	--	*N->C*	--
S	*N->S*	*S->R* *S->R*	*S->R*	*S->C* *C->S*	--
R	--	*S->R*	--	--	--
C	*N->C*	*S->C* *C->S*	--	*C->L* *C->L*	*C->L*
L	--	--	--	*C->L*	--

Table 4: NSRCL Interaction

The notation used in this Table is identical to that used in Table 3 above. Note the interaction when an active supporter meets an active contrarian. The outcome of this meeting (**C** becomes ***S***, ***S*** becomes ***C*** or no change) is determined by the probabilities of conversion, which can be thought of as the relative effectiveness of each group's

propaganda. Note also that when either the parameters or initial conditions are such that the group ***C*** does not come into play, this model collapses to that proposed by Daley and Kendall (1964).

The system of differential equations for this model follow:

$$
\begin{aligned}
&\frac{dN}{dt} = -\beta_1 NS - \beta_2 NC \\
&\frac{dS}{dt} = \beta_1 NS - \beta_3 S^2 - \beta_4 SR + \beta_5\ SC \\
&\frac{dR}{dt} = \beta_3 S^2 + \beta_4 SR \\
&\frac{dC}{dt} = \beta_2\ NC\ - \beta_5 SC - \beta_6 C^2 - \beta_7 CL \\
&\frac{dL}{dt} = \beta_6 C^2 + \beta_7 CL \\
&\text{with initial conditions:} \\
&N_0 = P\left(1 - \left(r_S + r_C\right)\right) \\
&S_0 = r_S P \\
&R_0 = 0 \\
&C_0 = r_C P \\
&D_0 = 0 \\
&r_I + r_C \le 1
\end{aligned}
\qquad (5)
$$

The definitions for the parameters in this model are similar to those for equations (1). Note that all β_i's are nonnegative except for β_5, which is positive if the active supporters are more effective than the active contrarians, and negative otherwise.

C. CLOSED FORM SOLUTIONS: THE NSC RECIRCULATION MODEL

In this section we present two models which are special cases of (2) when there are no latent classes; both active supporters (***S***) and active contrarians (***C***) may become neutral (***N***). We call these models *recirculation models* because there are no absorbing states; an individual may change classes infinitely many times. These models, labeled NSC, consider the flow of public opinion as opposed to ideology.

Both NSC models asymptotically converge to closed form solutions. For a detailed description of the methods used in this section, see Chapter 9 of Boyce and DiPrima (1997).

In this section we are adjusting our notation slightly. Here p_is represent recruitment coefficients into the ***S*** and ***C*** classes from ***N***, while β_is represent attrition coefficients – rates of flow of ***S*** and ***C*** back to ***N***. Additionally, we fix $P = N + S + C = 1$.

1. Quadratic Model

As with the *NSRC* and *NSRCD* models we assume that the attrition of the active spreaders is quadratic. The system of ODEs representing this model is:

$$\begin{aligned} \frac{dN}{dt} &= -p_1 NS - p_2 NC + \beta_1 S^2 + \beta_2 C^2 \\ \frac{dS}{dt} &= p_1 NS - \beta_1 S^2 \\ \frac{dC}{dt} &= p_2 NC - \beta_2 C^2 \end{aligned} \tag{6}$$

This model has the interpretation that persons become politically active for a time, and then, after their period of activity, they simply stop caring and meld back into the neutral population again. Note that the decay term for both the supporters and the contrarians is quadratic. The quadratic term represents the following phenomenon: In a large population, the more individuals are politically active on a given side, the less each individual feels he/she must contribute. In other words, the rate at which an active opinionated individual (supporter or contrarian) becomes neutral depends on how many people are active on his side; the more active people, the more an individual would be inclined to cease his activity.

The system of ODEs in (3) has a single asymptotic stable point which is given as a function of the model parameters, for non-trivial initial conditions. By asymptotic stable point, we mean that as time approaches infinity, the size of each class will converge to a single point.

The stable point of this system can be found by setting the derivatives equal to zero and solving algebraically. Provided that none of the β_i's are equal to zero and the initial conditions are non-trivial (there are initially active supporters and active contrarians), this system has a stable point at:

$$N_\infty = \frac{1}{1+\frac{p_1}{\beta_1}+\frac{p_2}{\beta_2}}$$
$$S_\infty = \frac{\frac{p_1}{\beta_1}}{1+\frac{p_1}{\beta_1}+\frac{p_2}{\beta_2}} \tag{7}$$
$$C_\infty = \frac{\frac{p_2}{\beta_2}}{1+\frac{p_1}{\beta_1}+\frac{p_2}{\beta_2}}$$

A trace of the system for given parameters and initial conditions $N_0 = .09, S_0 = .01, C_0 = .9$ is shown in Figure 6. All plots in this Chapter are made with MATLAB. See Section D for details.

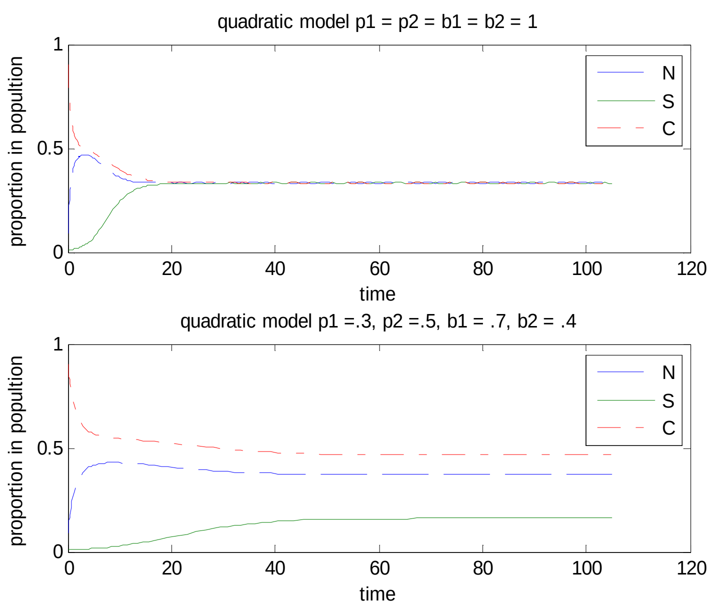

Figure 4. Plots of the NSC Quadratic model

The numerical computations agree with our theoretical asymptotic results.

2. Linear Model

In the second model we assume a constant attrition rate; fixed proportions of supporters and contrarians become neutrals. The system of ODEs representing this model is:

$$\begin{aligned}\frac{dN}{dt} &= -p_1 NS - p_2 NC + \beta_1 S + \beta_2 C \\ \frac{dS}{dt} &= p_1 NS - \beta_1 S \\ \frac{dC}{dt} &= p_2 NC - \beta_2 C\end{aligned} \qquad (8)$$

The linear nature of this model makes each individual's political activity period independent of the number of active individuals of his side.

This model requires a different approach than the quadratic one. We will assume non-trivial initial conditions and proceed by taking cases on the parameter values.

Case 1: $\beta_i > p_i, i = 1,2$.

In this case the attrition coefficients are, respectively, greater than the recruitment coefficients. In layman's terms, there is a net outflow. Next, we consider the equation corresponding to the supporters. The case of the contrarians (third equation) is treated similarly. At stability we have,

$$\frac{dS}{dt} = S(Np_1 - \beta_1) = 0 \qquad (9)$$

Because, by definition, $0 \le S \le 1$, the sign of the derivative is determined by the quantity $(Np_1 - \beta_1)$. Because $0 \le N \le 1$, this quantity is always negative in this case because $\beta_1 > p_1$. Therefore, for any $S > 0$, $\frac{dS}{dt} < 0$ and therefore the S population is decreasing to zero. An identical argument holds for the C equation.

From this discussion we conclude that the trajectory of the system is always pointed towards the point

$$S_\infty = 0, C_\infty = 0, N_\infty = 1 \qquad (10)$$

and that this is the stable point.

Figure 5 depicts the trace of the system for Case 1.

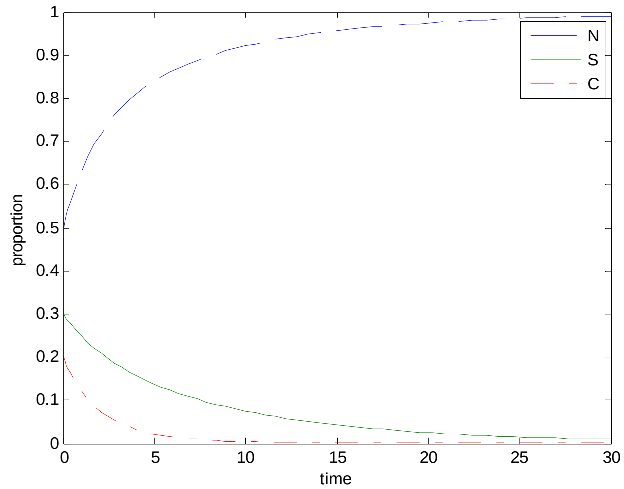

Figure 5. NSC linear model for Case 1

In Figure 5, $S_0 = .5, I_0 = .3, C_0 = .2, p_1 = .2, \beta_1 = .3, p_2 = .5, \beta_2 = .8$

Case 2: $\beta_1 > p_1 \ (\beta_2 > p_2); \beta_2 < p_2 \ (\beta_1 < p_1)$

We will mirror our analysis in Case 1 and examine the case where $\beta_1 > p_1$, $\beta_2 < p_2$, realizing that the result is symmetrical for the other case.

Employing the same reasoning as in Case 1, it follows that the *S* population decreases to zero. Therefore, the resulting stable point is only a function of the *N* and *C* populations, and comes from solving the system of equations:

$$C_\infty(\beta_2 - N_\infty p_2) = C_\infty(N_\infty p_2 - \beta_2) = 0 \tag{11}$$

If $C_\infty = 0$, then the last equation in (5), and the fact that $C_0 > 0$, imply that for *t* large enough $(p_2 N - \beta_2) < 0$. But as *t* gets larger $N \to 1$. Since $\beta_2 < p_2$ it follows that $(p_2 N - \beta_2) \geq 0$, in contradiction. Therefore $C_\infty \neq 0$ and we can solve (9) algebraically to obtain:

$$N_\infty = \frac{\beta_2}{p_2}, C_\infty = 1 - \frac{\beta_2}{p_2}, S_\infty = 0 \tag{12}$$

By symmetry, the case where $\beta_2 > p_2$, $\beta_1 < p_1$ has the solution:

$$N_\infty = \frac{\beta_1}{p_1}, C_\infty = 0, S_\infty = 1 - \frac{\beta_1}{p_1} \tag{13}$$

We present a trace of this system as Figure 6.

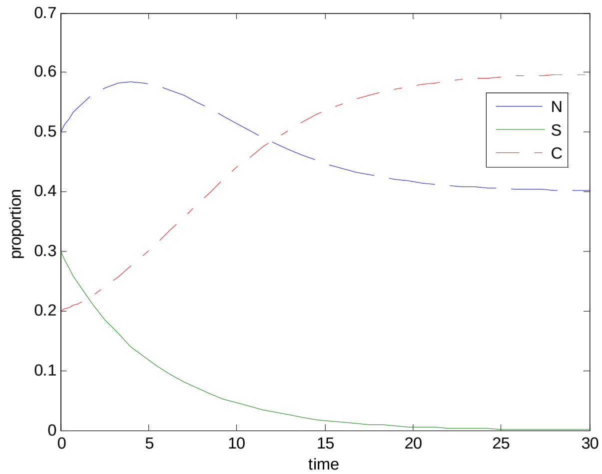

Figure 6. NSC linear model for Case 2

In Figure 6, $N_0 = .5, S_0 = .3, C_0 = .2, p_1 = .2, \beta_1 = .3, p_2 = .5, \beta_2 = .2$

Case 3: $\beta_i < p_i, i = 1, 2$

This case has two sub cases:

(i) $\dfrac{\beta_1}{p_1} \neq \dfrac{\beta_2}{p_2}$

And

(ii) $\dfrac{\beta_1}{p_1} = \dfrac{\beta_2}{p_2}$

<u>Case 3i</u>

It is immediate that $S_\infty \times C_\infty = 0$, since otherwise $N_\infty = \dfrac{\beta_1}{p_1}$ *and* $N_\infty = \dfrac{\beta_2}{p_2}$, which is a contradiction. Suppose that:

$$\frac{\beta_1}{p_1} < \frac{\beta_2}{p_2} < 1. \tag{14}$$

If $S_\infty = C_\infty = 0$ then $N_\infty = 1$, and therefore, from the first equation in (5), it follows that for t large enough $S(\beta_1 - Np_1) + C(\beta_2 - Np_2) > 0$ must hold. But for t large enough $N \to 1$ and therefore $(\beta_i - Np_i) < 0, i = 1, 2$, in contradiction. Hence we conclude that one of the following solutions is correct:

(a) $N_\infty = \dfrac{\beta_2}{p_2}, C_\infty = 1 - \dfrac{\beta_2}{p_2}, S_\infty = 0$

(b) $N_\infty = \dfrac{\beta_1}{p_1}, C_\infty = 0, S_\infty = 1 - \dfrac{\beta_1}{p_1}$

Suppose (a) is true. Let $\varepsilon > 0$ be such that

$$\varepsilon < \left[\frac{\beta_2}{p_2} p_1 - \beta_1 \right] \frac{1}{p_1} \tag{15}$$

The right-hand-side of (13) is positive because $\dfrac{\beta_1}{p_1} < \dfrac{\beta_2}{p_2}$.

Let *t*’ be large enough such that

$$\left| N - \frac{\beta_2}{p_2} \right| < \varepsilon \tag{16}$$

For $t \geq t'$. Such a t' exists because $N \to \dfrac{\beta_2}{p_2}$.

From (13) and (14) it follows that

$$Np_1 - \beta_1 > 0 \text{ for } t \geq t'. \tag{17}$$

Let $t'' \geq t'$ such that

$$S(t'') > 0 \tag{18}$$

and

$$S'(t'') < 0 \tag{19}$$

Such t'' exists because $S_0 > 0$ and $S \to 0$.

From (15) and (16) we obtain that $S'(t'') > 0$, which contradicts (17).

Therefore (b) is the correct solution for case 3i.

By symmetry, for the case where $\frac{\beta_2}{p_2} < \frac{\beta_1}{p_1} < 1$ the solution is given by:

$$N_\infty = \frac{b_2}{p_2},\ I_\infty = 0, C_\infty = 1 - \frac{b_2}{p_2} \tag{20}$$

We present a trace of Case 3i as Figure 7:

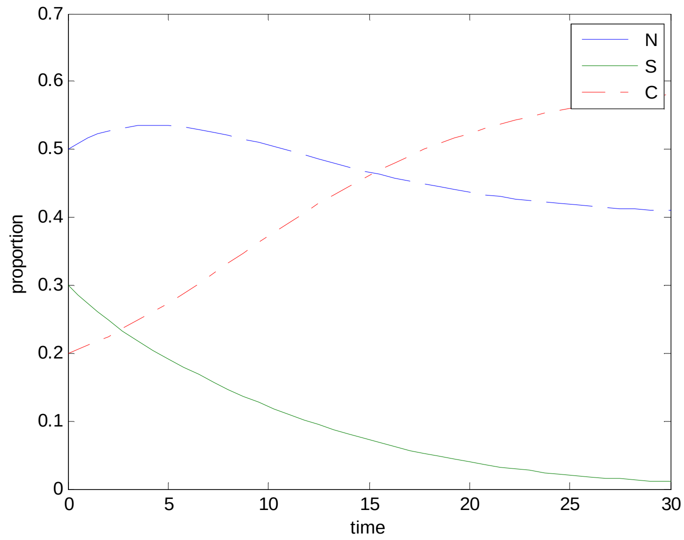

Figure 7. NSC Linear model for Case 3(i)

In Figure 7, $N_0 = .5, S_0 = .3, C_0 = .2, p_1 = .2, \beta_1 = .3, p_2 = .5, \beta_2 = .8$

<u>Case 3ii</u>

From the second and third equations of (5) we have that

$$N = \frac{S'}{Sp_1} + \frac{\beta_1}{p_1} = \frac{C'}{Cp_2} + \frac{\beta_2}{p_2} \tag{21}$$

Because $\frac{\beta_1}{p_1} = \frac{\beta_2}{p_2}$, it follows that (19) is equivalent to the differential equation:

$$p_2[\log S]' = p_1[\log C]'. \tag{22}$$

Since S_∞ and C_∞ cannot be both zero, it follows that

$$N = \frac{b_1}{p_1} = \frac{b_2}{p_2} \tag{23}$$

and from (19) and (20) we obtain S_∞ as a solution of:

$$\frac{S_\infty^{\ p_2}}{\left(1 - \frac{\beta_1}{p_1} - S_\infty\right)^{p_1}} = \frac{S_0^{\ p_2}}{C_0^{\ p_1}} \tag{24}$$

Since $N_\infty + S_\infty + C_\infty = 1$, we obtain also a value for C_∞. We present a trace of case 3(ii) as Figure 10.

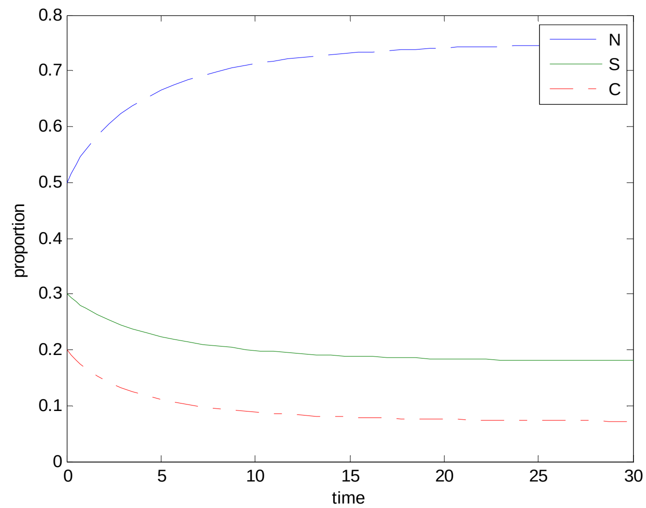

Figure 8. NSC linear model for Case 3(ii)

In Figure 8, $N_0 = .5, S_0 = .3, C_0 = .2, p_1 = .4, \beta_1 = .3, p_2 = .8, \beta_2 = .6$

D. NUMERICAL ANALYSIS

1. Numerical Methods

In Section C we developed closed-form solutions to some special cases of the general models. Analytic solutions to the cases that we are most interested in, namely NSRL and NSRCL, are more elusive. We proceed with numerical analysis of these two models.

The fact that we do not possess a closed-form solution to the cases of interest to us is not as great a difficulty as it may initially seem. We may approximate the solution to the ODE system numerically. For solving the two aforementioned models and perform the analysis we use the routine **ode45** supplied with the software package MATLAB. The basis of this routine is the 4th order Runge-Kutta method. Interested readers are directed to Chapter 8 of (Boyce and DiPrima, 1992).

For the **NSRL** model we assume a *base case* where all the transmission coefficients β_i are equal to 1. As before, we assume that P =1.

2. The NSRL Model

For the **NSRL** model we assume a *base case* where all the transmission coefficients β_i are equal to 1. As before, we assume that P =1. We first present in Figure 9 the *base case* trace in time with $r_S = r_L = .1$.

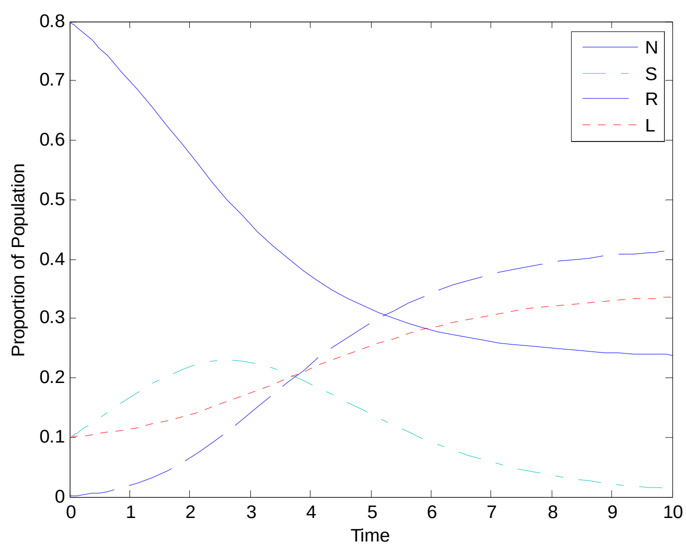

Figure 9. NSRL model: base case

Notice that after 10 time units the active supporters are about 10% of their initial number. The proportions in the population of latent supporters (***R***) and latent contrarians (***L***) converge in the long run to .42 and .34, respectively.

We next investigate the effect of the initial sizes of the ***S*** (active supporters) and ***L*** classes on the composition of the population at stability. Specifically, for a given initial proportion r_S of active supporters we ask: what is the initial proportion of latent contrarians r_L such that at stability the number of latent supporters R_∞ is equal to the number of latent contrarians L_∞? This Measure of Effectiveness (MOE) indicates the "neutralization" effect of the two classes of population – the initial conditions that lead to parity. Many other MOEs, including time-dependant ones, could be developed and implemented.

Clearly, the final ratio between the supporters and the contrarians $\frac{R_\infty}{L_\infty}$ is a monotone decreasing function of the initial fraction r_c of contrarians.

We use a binary search in MATLAB to iteratively approach the critical proportion of initial contrarians for each initial proportion of pro-spreaders. This is a technique we will use frequently in the following analyses. We set the effectiveness parameters at their base case values. Figure 12 presents the proportion of initial latent contrarians r_L required for achieving parity for a given proportion r_S of initial supporters.

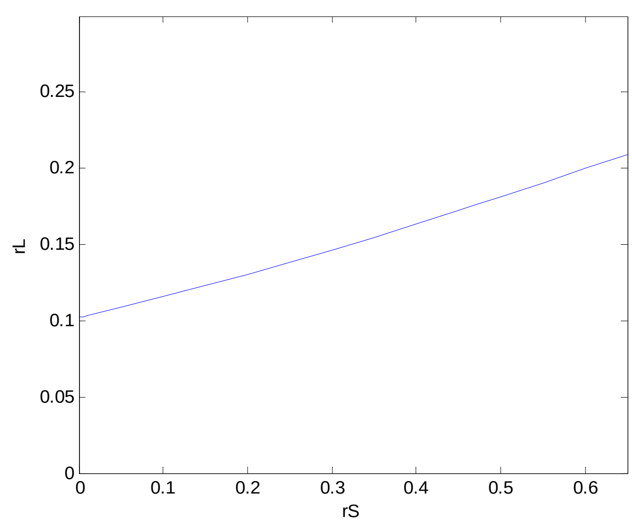

Figure 10. Proportion of initial latent contrarians r_L required for achieving parity for a given proportion r_S of initial supporters.

Figure 10 is interesting because it shows that for the given modeling assumptions, a somewhat large increase in the ***S*** population is offset by a modest increase in the ***L*** population.

Recall that in Figure 9, the ***N*** population (neutrals) does not go to zero as time goes to infinity, but rather reaches a stable point. We next consider the size of the ***N*** class at parity as a function of the initial proportion of supporters r_S. Note that the parity condition uniquely determines the value of r_L, as shown in Figure 10.

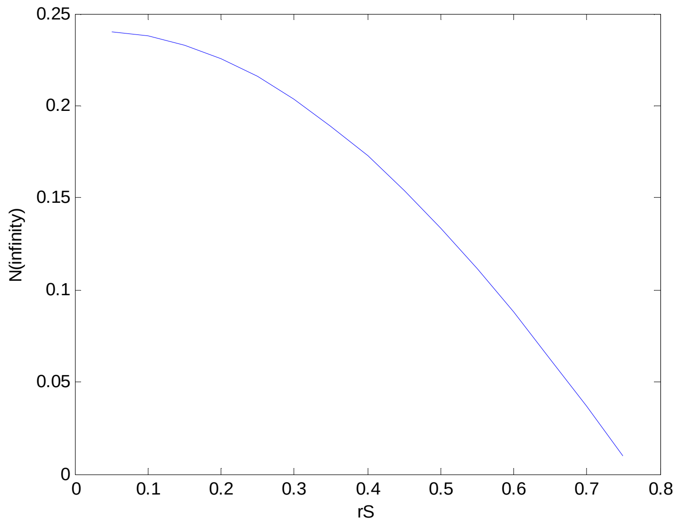

Figure 11. N(infinity) at parity for increasing numbers of initial supporters

Figure 11 shows that the number of neutrals approaches zero rather rapidly at parity, as the initial number of supporters increases. A surprising result from this graph is that the introduction of class ***L*** causes ***N***(infinity) to be larger than .203, which is the expected proportion from Daley and Kendall (1964). We attribute this to the fact that the attrition of class ***S*** is faster in our model than in theirs, making fewer available to recruit out of class ***N***.

We next consider the implications of varying the effectiveness parameter of the latent contrarians, β_4. We vary the parameter β_4 and find the value of the parameter r_L that is required to achieve parity. We fix $r_S = .1$ throughout. Figure 14 shows the result of this experiment.

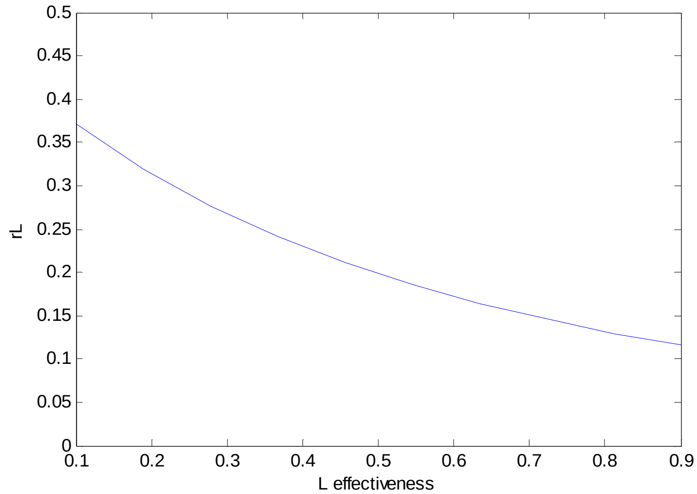

Figure 12. Critical r_L with varying effectiveness β_4, r_S =.1

Note in Figure 12 that as β_4 approaches 1, r_L approaches .102, which is the value shown in Figure 12 for r_S =.1.

We next consider the behavior of the model when we vary the parameters β_1 and β_4, which are the ***S*** recruitment rate from the ***N*** population and the ***L*** conversion rate from the ***S*** populations, respectively. We fix the values of r_S and r_L at .1 each (as in the base case) and search for parameter values that achieve parity.

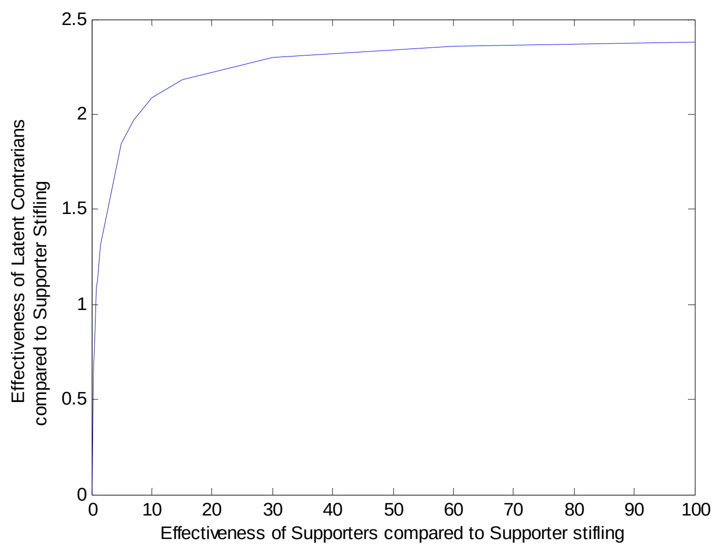

Figure 13. Contrarian effectiveness required to achieve parity, $r_S = r_L$ =.1.

Note that as the conversion effectiveness increases, a very small increase in the effectiveness of the contrarians must be countered by a very large increase in the effectiveness of the active supporters in order to attain parity. For example, when $\beta_4 = 1$, then $\beta_1 = 0.75$ at parity. However, when $\beta_4 = 2$, then $\beta_1 = 7.5$ at parity, and $\beta_4 = 2.2$, results in $\beta_1 = 15$.

Our numerical analysis has focused on the *scaled* version of the process. We now consider the implications of using an un-scaled process, allowing the population size ***P*** to grow while leaving the number of initial active supporters $\boldsymbol{S_0}$ fixed. This mirrors the analysis performed by Daley and Kendall (1964). In Figure 16, we fix the number of initial spreaders at 1, and allow the population size to grow.

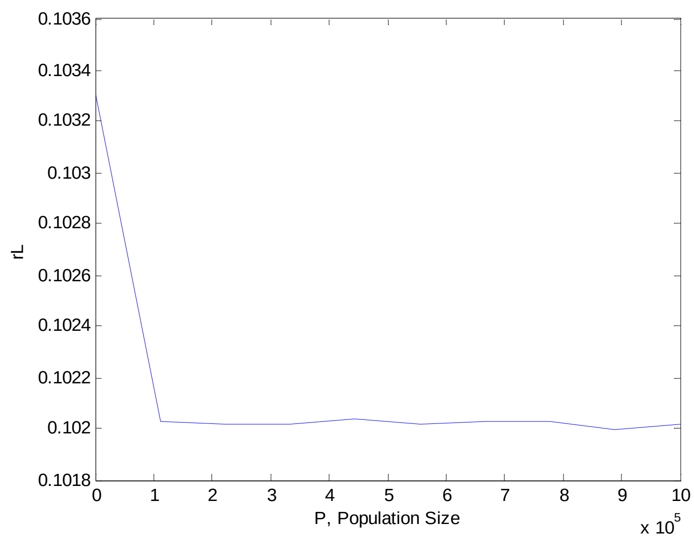

Figure 14. Critical r_L , $S_0 = 1$

As the population increases, the proportion of contrarians required to achieve parity appears to be invariant to the population size. The source of the variability in Fig. 14 is an artifact of double-precision arithmetic and not a property of the system itself.

The deterministic model has the behavior that the final distribution of classes ***N, R***, and ***L*** is invariant to a fixed S_0 for large populations. Stochastic models do not exhibit this behavior and is an example of a case where deterministic models are unsatisfactory for analysis. We address this issue in the following chapter.

3. NSRCL

Next we consider the symmetric model, which has more variables than the NSRL model. For the **NSRCL** model we assume a *base case* where all the transmission coefficients β_i are equal to 1 *except* $\beta_5 = 0$ (recall that β_5 is governs the 'conversion' of active members of the pro- and counter- classes, an asymmetric behavior). As before, we assume that P =1.

Figure 17 below presents the trace of the model for the base case.

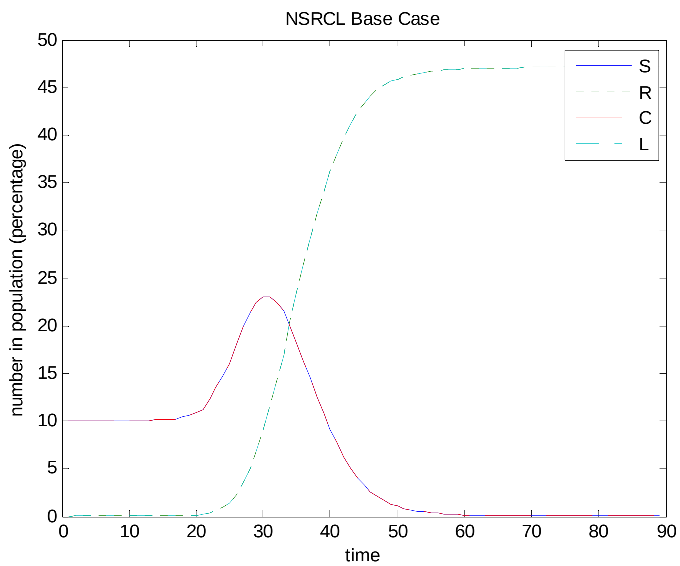

Figure 15. NSRCL Base Case

Note that due to the symmetric nature of the model, the traces of S and C as well as R and L are coincident.

We next present four time traces of the NSCRL model, slightly perturbed from the Base Case, in order to show the behavior of the population classes (Class ***N*** is not shown for the sake of clarity). We show these plots as a contrast to Figure 15, showing the behavior of the classes under non-homogeneous parameter values.

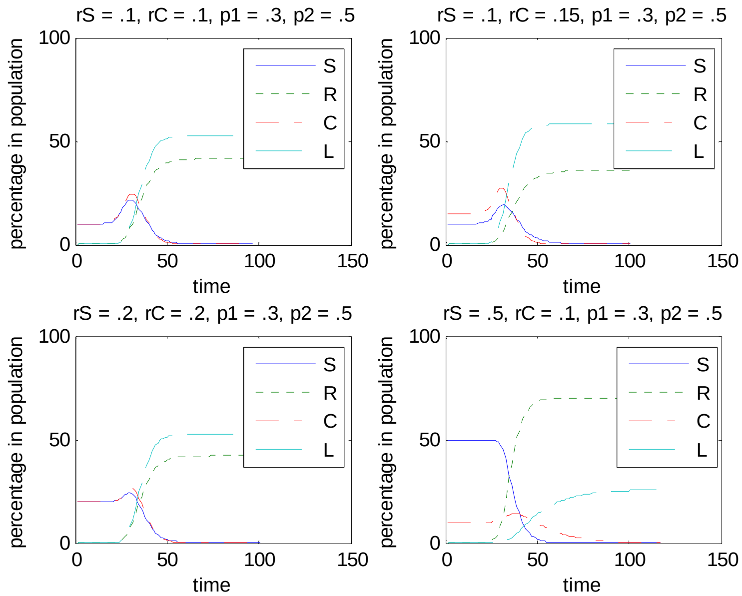

Figure 16. NSRCL Model, with varying initial conditions

Similar to our analysis of the NSRL model, we next consider the proportion of spreaders of each type to achieve parity in the base case (all conversion intensities β_i equal to 1).

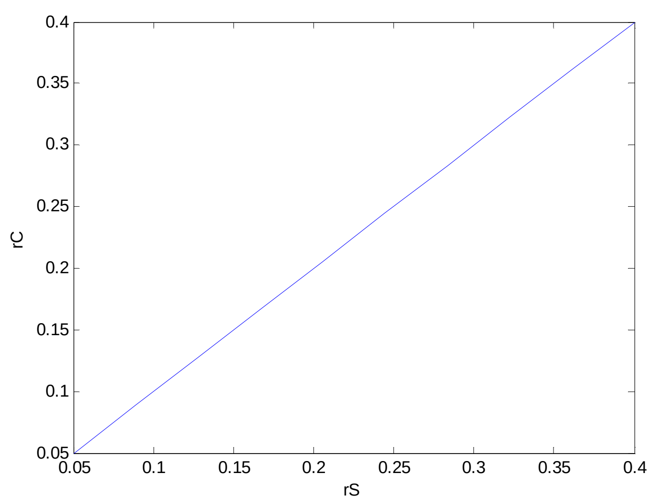

Figure 17. Proportion of active contrarians r_c to achieve parity for a given initial proportion of active supporters.

As the NSRL model is symmetric, the linear plot in Figure 17 is not surprising. We repeat the analysis for a slightly different case. Here we fix the parameter β_2 at .5, making the contrarians half as effective at recruitment from the neutral population as the supporters.

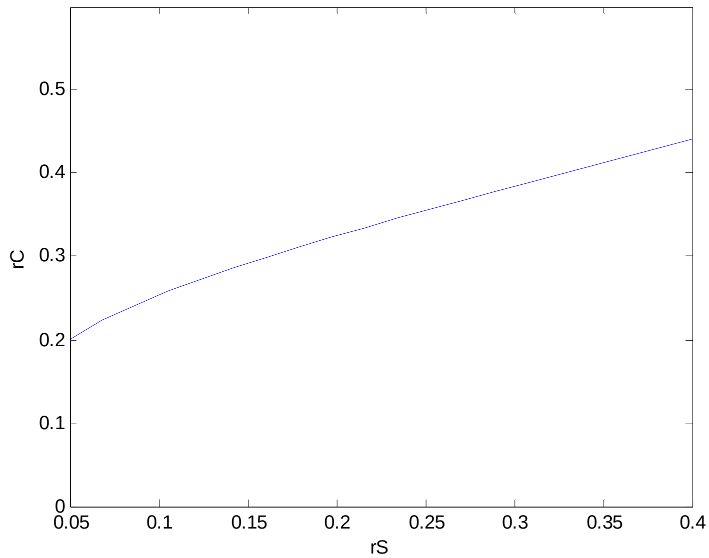

Figure 18. Proportion of active contrarians r_c to achieve parity for a given initial proportion of active supporters. $\beta_2 = .5$

From Figure 18 we see that the effect of lower contrarian recruitment effectiveness depends on the proportion of supporters r_S; for small values of this parameter the critical proportion of initial contrarians r_C is considerably higher than r_S, while for larger values of r_S the behavior is similar to the case of $\beta_2 = 1$ shown in Figure 19.

We conclude our analysis with respect to β_2 by fixing the initial proportion of supporters at its base case (r_S= .1) and varying the effectiveness of recruitment by the contrarians β_2. Figure 19 presents the results of this analysis.

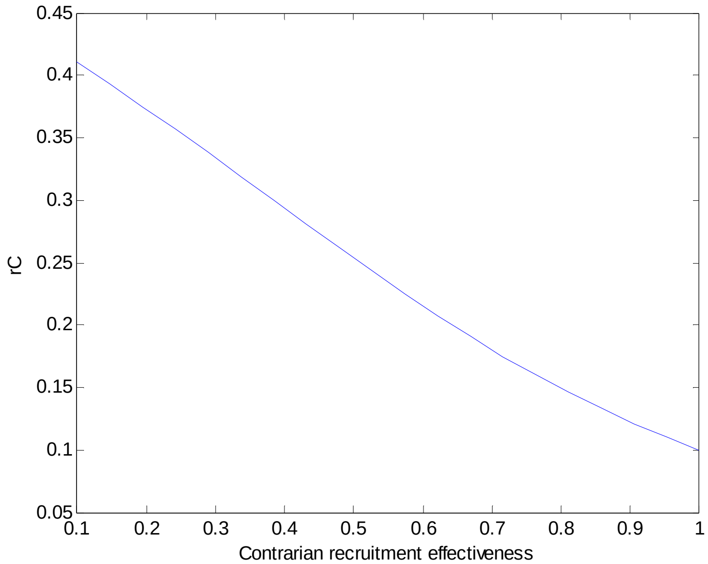

Figure 19. Critical proportion of contrarians to achieve parity for varying contrarian recruitment effectiveness β_2 . r_S=.1

The results in Figures 18 and 19 are surprising at first; we might expect parity to be unachievable for low recruitment effectiveness. The explanation for this phenomenon is that the ability to recruit is offset by having such a large proportion of the population on the counter side initially.

We repeat the analysis in Figure 19, but instead of varying β_2, we set $\beta_6 = \beta_7$ and vary these parameters together. Recall that these parameters govern the rate of conversion for the contrarians, ***C***, into the absorbing state ***L*** both via ***C-C*** interaction and ***C-L*** interaction**.**

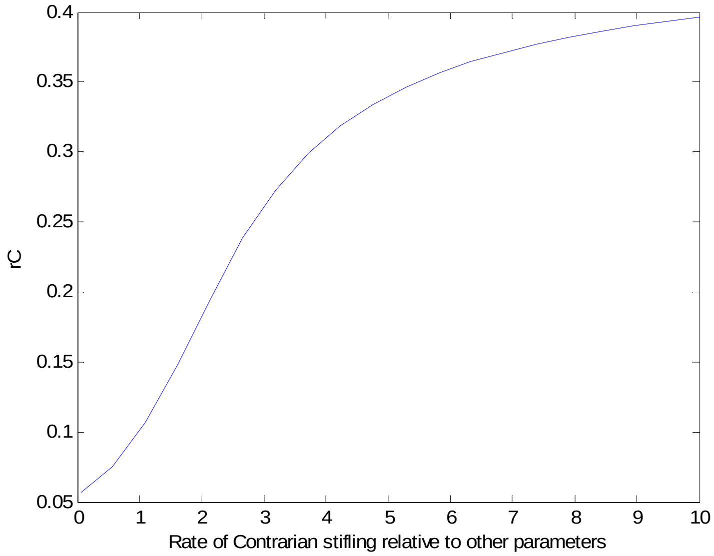

Figure 20. Critical proportion of Class C to achieve parity, varying $\beta_6 = \beta_7$, all other parameters as in the base case

An interpretation of varying $\beta_6 = \beta_7$ is that for small values, the active life of an active Contrarian is very long; for large values, it is very short. We see here that as their life span increases towards infinity, the proportion required to achieve parity becomes very small. As their lifespan becomes very short (large values of $\beta_6 = \beta_7$), the proportion required approaches .4, which is to say that their recruitment is virtually nil.

The NSRCL model has no invariance property analogous to the behavior shown in Figure 14.

The final issue we consider is the effect of varying the number of initial supporters, initial contrarians, and the effectiveness of conversion. Recall that β_5 is the rate at which supporters 'convert' contrarians. This is the only parameter in our model that is allowed to take on negative values (a negative value means that contrarians convert supporters).

We vary the parameter β_5 along the X axis of Figure 22; we vary r_C along the Y axis. The z axis is $L_\infty - R_\infty$. We fix $r_S = .1$ throughout.

Because the model is symmetric, the same results would be obtained if everything were reversed; that is, what we consider to be supporters and contrarians in this analysis is unimportant as long as we are consistent. In other words, we could have just as well varied r_S and β_5; the plot obtained would be a mirror image of the ones shown below.

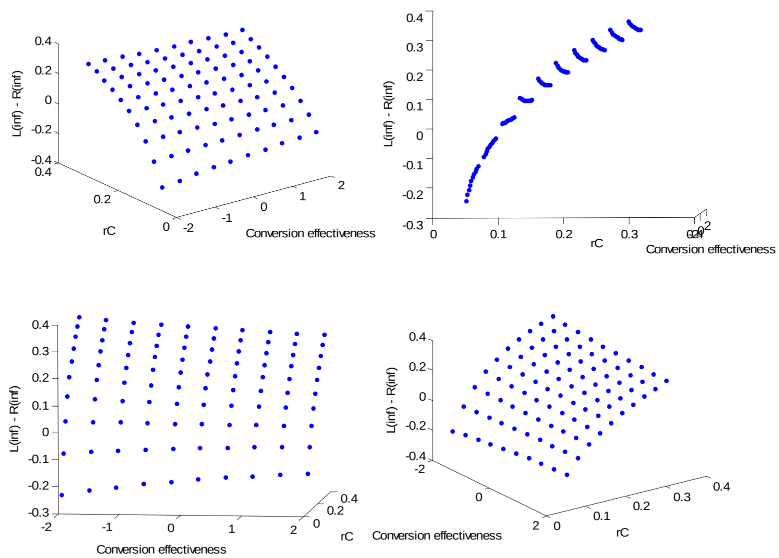

Figure 21. The numerical difference between L_∞ and R_∞ as a function of model parameters (four views)

Figure 21 presents four views of the same plot. We see from looking at the plot oriented along the r_C axis that in the ranges considered here, the initial conditions have a greater effect than the effectiveness of conversion.

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IV STOCHASTIC MODELS

A. MOTIVATION

While in Chapter III we presented a deterministic approach to the ideological propagation problem, in this chapter we obtain additional insights by studying the stochastic process that describes the ideological propagation among different population categories. The main benefit of stochastic models as opposed to deterministic ones is that they enable the analysis of random interactions among individuals; the cost is that the resulting models become less tractable.

Because NSC models lack absorbing states, the techniques employed in this chapter are not appropriate for the NSC recirculation models introduced in Chapter III. In this chapter we focus on the asymmetrical model (NSRL). Recall that the NSRL model has four different types of actors:

- ***N*:** the neutrals, who are members of the population who carry neither the pro- or counter- message.
- ***S***: the active supporters, who carry the pro- message and are actively spreading it.
- ***R***: the retired supporters, who carry the pro- message but are not spreading it.
- ***L***: the latent contrarians, who carry the counter- message but are not actively spreading it.

The engagement rules among these groups are the same as those proposed in Table 3 of Chapter III. One difference is that the parameter η in our stochastic development takes the place of the parameter β_4 in the deterministic model; η describes the effectiveness of the Latent Contrarians ***L*** in converting members of class ***S***. We choose this as the single degree of freedom in the parameters.

The engagement rules described in Table 3 imply that the process is strictly evolutionary, so that an absorbing state is reached in finite time with probability one. This fact indicates that capturing the main properties of the distribution of the state of the process when it enters the absorbing states – the states from which no further dynamics occur - is crucial towards understanding the implications of our models. This is the focus in this chapter.

This chapter is organized as follows: In section B we formulate our model. Section C presents our results for the small population scenario, while Section D does the same but for large populations. Section E presents the conclusions of this Chapter.

B. MODEL FORMULATION

In this section we formulate the continuous time Markov chain and the associated discrete time chain that form the core of our model.

A population of size P is split among 4 mutually exhaustive groups: Neutrals (**N**), Supporters (**S**), Retired Supporters- (**R**) and Latent Contrarians (**L**). We assume that the time dynamics of these groups follow a continuous time Markov chain

$$((N,S,R,L)(t):t\geq 0), \tag{1}$$

with state space

$$\mathbf{S}=((n,s,r,l)\in Z^4 : n+s+r+l=P, n\geq 0, s\geq 0, r\geq 0, l\geq 0), \tag{2}$$

and initial conditions

$$(N,S,R,L)(0)=\left(P-S_0-L_0,S_0,0,L_0\right), \tag{3}$$

where S_0 and L_0 are non-negative integers. In our notation, *N*(t) is the number in class ***N*** at time *t*; *S*(*t*), *R*(*t*) and *L*(*t*) are likewise.

Given that for all $t\geq 0$ we have $R(t)=P-N(t)-S(t)-L(t)$, much of what follows is without loss of generality framed in terms of the $(N,S,L)(.)$ process, which for notational convenience we define as $\mathbf{X}=((N,S,L)(t):t\geq 0)$. The transition rates are given by

$$(n,s,l)\rightarrow(n\text{-}1,s+1,l) \text{ at rate } \beta ns \tag{4}$$

for the *N*-*S* interaction, and by

$$(n,s,l)\rightarrow(n,s-2,l) \text{ at rate } \beta\binom{s}{2}. \tag{5}$$

for the *S*-*S* interaction. For the *S*-*R* interaction the transition rate is

$$(n,s,l) \rightarrow (n,s-1,l) \text{ at rate } \beta s(P-n-s-l).$$

The *S-L* interactions are governed by the transition rate

$$(n,s,l) \rightarrow (n,s-1,l+1) \text{ at rate } \beta\eta sl. \qquad (6)$$

The parameter β is the stochastic mixing rate parameter and $\eta, 0 \le \eta \le 1$, is the effectiveness of the contrarians. Figure 22 illustrates the possible transitions out of state (n,s,l).

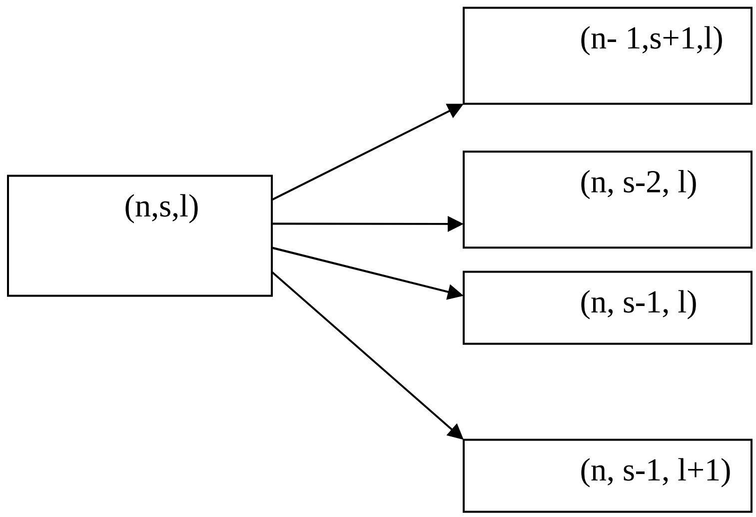

Figure 22. State Transitions

The evolutionary nature of the process $\mathbf{X}$, as evidenced by its transition rates, implies that $\mathbf{X}$ reaches an absorbing state in finite time with probability 1. Hence, in order to study the distribution of the absorbing state it suffices to consider the associated discrete time Markov chain (DTMC) of $\mathbf{X}$. For this reason let

$$\mathbf{Y}_n = \left((N,S,L)_n : n \ge 0\right) \qquad (7)$$

be a discrete time Markov chain defined on the same state space and with the same initial conditions as $\mathbf{X}$. Due to finite population, for all $n \ge 0$ the relationship

$$R_n = P - N_n - S_n - L_n \qquad (8)$$

holds, so we know the amount of the retired pro spreaders by storing information about only three groups. We now consolidate equations (1-8) and derive the transition matrix for **Y**:

$$(N,S,L)_{n+1} = (N,S,L)_n +$$

$$\left\{\begin{array}{lll} (-1,+1,0) & w.p.\ \dfrac{n}{f_{nsl}} & (a) \\ (0,-2,0) & w.p.\ \dfrac{\frac{1}{2}(s-1)}{f_{nsl}} & (b) \\ (0,-1,0) & w.p.\ \dfrac{P-n-s-l}{f_{nsl}} & (c) \\ (0,-1,+1) & w.p.\ \dfrac{\eta l}{f_{nsl}} & (d) \end{array}\right\} \tag{9}$$

Where

$$f_{nsl} = P - \tfrac{1}{2}s + \eta l - l - \tfrac{1}{2} \tag{10}$$

The transition matrix is conditional on a transition occurring; we scale the rates of transition in equations (9) by the factor in equation (10) to have a matrix where the probabilities sum to 1. Additionally, the rate parameter β has divides out.

These transition rules indicate that the absorbing states of $\mathbf{Y}$ are given by

$$\mathbf{B} = ((n,s,l) \in \mathbf{S} : s = 0)\,, \tag{11}$$

and that $P(\tau < \infty) = 1$, where

$$\tau = \inf\{n \geq 0 : \mathbf{Y}_n \in \mathbf{B}\} \tag{12}$$

Due to the evolutionary nature of the process, we can place a firm deterministic bound on the value of τ. Consider the limiting case on the initial conditions where the population consists of 1 ***S*** and *P*-1 ***N***. Each member of the susceptible population must pass through class **S** en-route to an absorbing state. Now consider a (possible but unlikely) limiting case where $N_\infty = 0$. Each member of the susceptible population must pass through class **S** en-route to an absorbing state, for $P-1 \approx P$ transitions. The maximum number of transitions out of class **S** occurs when S-S stifling does not occur.

This path is unlikely, but possible. If we consider the maximum number of transitions out of class **S**, we find it is P. Therefore, $\tau \leq 2P$. Note that this is a *limiting* case in the absence of contrarians. The addition of contrarians will necessarily reduce the maximum number of transitions further.

C. RESULTS FOR SMALL POPULATIONS

1. Algorithm

Equations (9) and (10) contain all the information about the process **Y,** but some care is required to extract useful information about the distribution of of the state of the process when it enters the absorbing states; that is the distribution of the number of individuals in classes R and L when the number of individuals in class S becomes 0. A naïve direct approach would involve mapping the three-dimensional state space onto one dimension, but it quickly becomes intractable for non-trivial population sizes, and the resulting transition matrix is very sparse, which leads to numerical issues due to insufficient computer memory.

A more refined approach that is numerically stable for population sizes in the order of the hundreds is to proceed sequentially, as described next.

Algorithm: Compute Absorbing Distribution

$Initialize:$

$PR_{(P+1)(P+1)(P+1)} = \mathbf{0}$

$TEMP_{(P+1)(P+1)(P+1)} = \mathbf{0}$

$Absorbing_{(P+1)(P+1)} = \mathbf{0}$

$while\left(any\, element\, of\;\; PR > 0\right)\{$

$TEMP = PR$

choose one element of PR to update{

Apply Eqn(9) to PR to determine successor state probabilities.

Add to $TEMP_{successor\ state}$

Set $TEMP_{current\ state} = 0$

$if\ (successor\ state\ is\ Absorbing)\{$

Add to $Absorbing_{successor\ state}$

Set $(TEMP_{successor\ state} = 0)$

}

$PR = TEMP$

}

Algorithm 1

The 3-dimentional matrix *PR* contains the probabilities associated with each non-absorbing state. It has dimension $P+1$ in order to include zero as a value for each population.

Each State (N, S, L) has up to four successor states and multiple predecessor states; a 'breadth-first' approach, such as above, is required. It is useful to think of the analogy of a wave moving out from a stone dropped in a pond. This is in essence how the probability mass travels from the (given) initial state. The non-zero elements of *PR* are on the 'crest' of the wave. The successor probabilities are computed step-wise; when the successors of any given state are computed, the wave has passed. Algorithmically, that state is no longer considered; its probability is assigned to zero. This process continues until all of the probability mass has diffused to absorbing states.

TEMP is merely a temporary copy of *PR*.

Absorbing is a $(P+1)\times(P+1)$ matrix, holding the probability mass associated with the number of individuals in classes R and L when the number of individuals in class S becomes 0, defined in equation (11). Note that because **S** is zero for the absorbing states, two dimensions are sufficient to contain the distribution of **X**.

At each jump epoch, the probability mass diffuses from each state, eventually reaching an absorbing state where it accumulates. When the process is 'near' a boundary (i.e. one jump from a disallowed state), some care must be exercised to ensure that the process remains in the non-negative integers. The only boundary rule that needs to be explicitly enforced is to disallow the **S** to **S** stifling process when only one spreader is active. We implement the above in MATLAB, with the code found in Appendix I.

2. Numerical Results and Analysis

We report the results of our computations with respect to the distribution of the number of individuals in classes R and L when the number of individuals in class S becomes 0. The only limit on the accuracy of the computation is the numerical precision of the computer. We show the plots below for four cases:

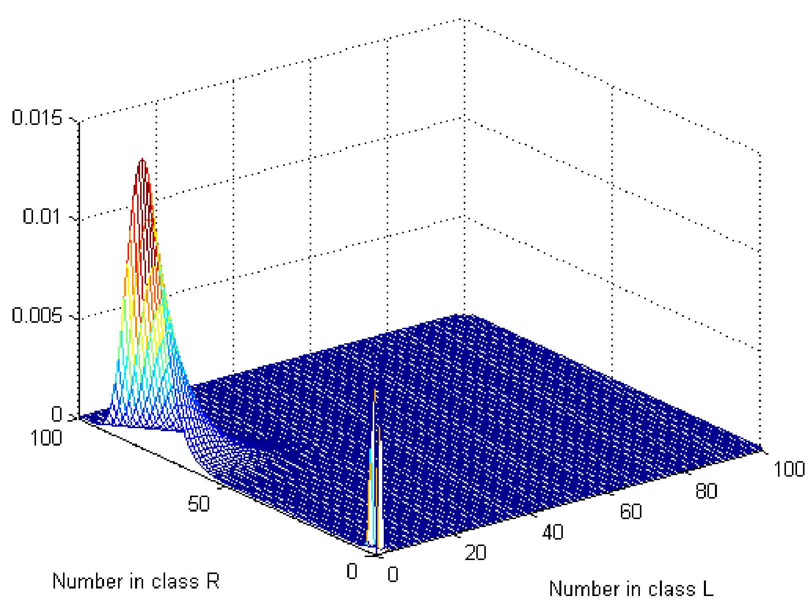

Figure 23. Case I: $P = 100,\ L_0 = 1,\ S_0 = 1,\ \eta = 1$

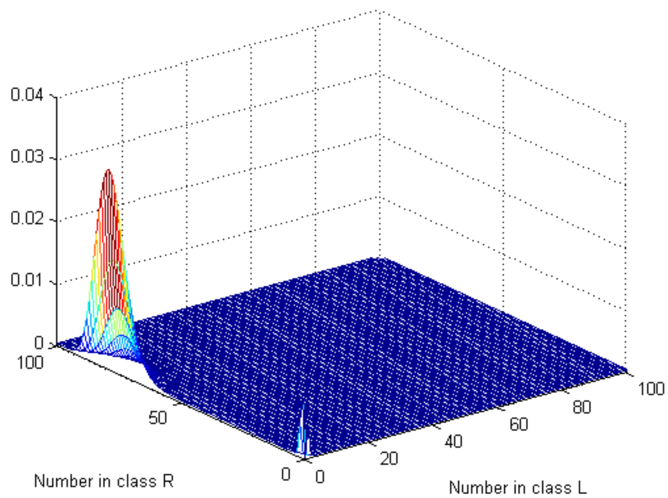

Figure 24. Case II : $P = 100,\ L_0 = 1,\ S_0 = 1,\ \eta = .5$

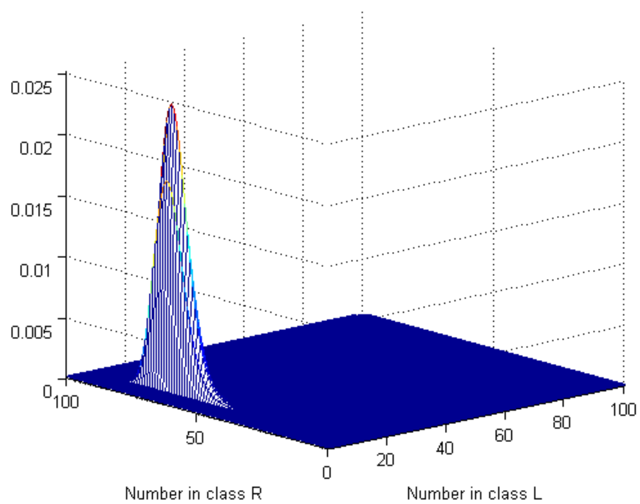

Figure 25. Case III: $P = 100,\ L_0 = 10,\ S_0 = 10,\ \eta = .1$

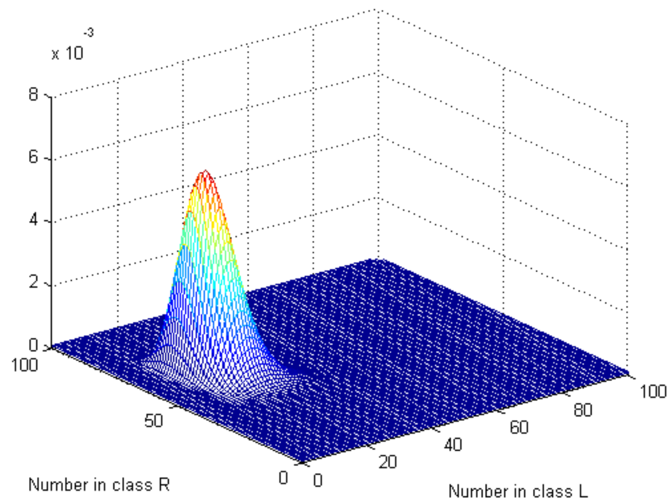

Figure 26. Case IV: $P = 100,\ L_0 = 10,\ S_0 = 10,\ \eta = 1$

In Figures 23 and 24 $S_0 = 1$ and there are positive limiting probabilities for the number of individuals in class R and class L being close to 0. The sum of these probabilities is the probability that the ideological spread 'fizzled out' and does not gain traction throughout the population as a whole; that is, the population remains neutral.

Figures 25 and 26 show cases where the process spreads through the population. These are in stark contrast to Figures 23 and 24, where there is a large 'spike' near the origin. This spike represents the probability that nothing significant happens.

The probabilities displayed in Figures 25 and 26 look characteristically 'bell-shaped' or Gaussian; in the next section we argue that they are approximately of that form.

D. RESULTS FOR LARGE POPULATIONS

The goal in this section is to study the distribution of the number of individuals in classes R and L when the number of individuals in class S becomes 0 for large population sizes, a situation where Algorithm 1 cannot be implemented due to memory constraints. As we shall momentarily see, the analysis of this section lends support to the apparent normality of the distribution of the number of individuals in classes R and L at the time the number of individuals in class S becomes 0 observed in Figures 25 and 26. The basis

for this section are the papers Barbour (1972) and Barbour (1973), who proves a functional Central Limit Theorem for Markov processes of the type described in this chapter, and provides an analytically tractable approach to solve the differential equations that arise in this setting. The focus on this section is on applying the results of these two papers to our propagation of ideas problem. The key step is to normalize the population size, and relax the requirement for an integer state space. In this section, we consider the proportions (non-integer) of the population in each class R and L as opposed to the (integer) members in each class.

1. Theoretical Results

With this in mind, consider a sequence of CTMC's $(\mathbf{X}_P)$ indexed by the population parameter *P*, and defined on $t \geq 0$. Each process $\mathbf{X}_P$ has initial condition

$$(N,S,L)(0) = \left(P(1-r_S-r_L), P\,r_S, P\,r_L\right) \tag{13}$$

for some constants $r_S > 0,\ r_L > 0,\ r_S + r_L < 1$. Because the initial size of each class grows with the population size, our initial conditions suggest that the propagation will not die out in its initial stages, as observed in Figures 23 and 24.

To complete the framework, let

$$\mathbf{x}_P(t) = P^{-1}\mathbf{X}_P(t) \ \text{ for } 0 \leq t \leq T \tag{14}$$

be a sequence of stochastic processes, where each $\mathbf{x}_P(t)$ represents the proportion of people in a population of size *P* that at time *t* belong to each class *N*, *S*, or *L*.

The transition rates for the scaled process $\mathbf{x}_P$ are given by

$$\begin{aligned}
(n,s,l) &\to \left(n-\frac{1}{P}, s+\frac{1}{P}, l\right) \text{ at rate } \beta nsP^{-1} \\
(n,s,l) &\to \left(n, s-\frac{2}{P}, l\right) \text{ at rate } \beta\binom{s}{2}P^{-1} \\
(n,s,l) &\to \left(n, s-\frac{1}{P}, l\right) \text{ at rate } \beta s(P-n-s-l)P^{-1} \\
(n,s,l) &\to \left(n, s-\frac{1}{P}, l+\frac{1}{P}\right) \text{ at rate } \beta\eta slP^{-1}
\end{aligned} \tag{15}$$

subject to the boundary conditions

$$n \geq 0,\ s \geq 0,\ l \geq 0 \tag{16}$$

It can be verified that $\mathbf{x}_P(t)$ meets the conditions needed to apply the results of Barbour (1972) and (1973). The result in these papers that is most useful for our purposes is that

$$\sup_{\upsilon} \left| P(Z_P(t) \leq \upsilon) - \Phi(\upsilon / \Sigma(t)) \right| = O\left(P^{-1/2} \log P\right) \tag{17}$$

for fixed *t*, $0 < t \leq T$, where:

$\Phi(.)$ is the distribution function of a standard normal variable;

$\Sigma(t)$ is a deterministic variable that depends on *t* and the parameters of the process;

$Z_P(t) = P^{1/2}\left\{\mathbf{x}_P(t) - \xi(t)\right\} \bullet h\left(\xi(t), t\right)$ is the normalized version of $\mathbf{x}_P(t)$, where

$\xi(t)$ is the deterministic solution found in Section 2 of Chapter III

h(.,.) is a deterministic vector-valued function that depends on the problem parameters;

T is the largest time, possibly infinite, for which we can find a deterministic solution utilizing the methods in Section 2 from Chapter III.

The result is that a normalized version of $\mathbf{x}_P(t)$ converges in functional space to a Normal random variable with variance that depends on the initial conditions and the rate parameters, as $P \to \infty$; in words, this means that the distribution of $Z_P(t)$ is never further than $O\left(P^{-1/2} \log P\right)$ from the Normal distribution. This result is operationally significant. First it implies that the transient regime and the state upon absorption of the stochastic model that describes the propagation of ideas is random. Secondly, for any time *t*, $0 < t \leq T$, the distribution of the proportion of individuals in each ideological group is approximately Normal with mean given by the deterministic solution, and covariance structure that can be analytically computed and depends on the initial conditions and rate parameters.

In the next subsection we illustrate these results.

2. Numerical Results and Analysis

In this subsection we present various illustrative scenarios. We start by comparing the mean of $Y_{\tau,\text{stochastic}}$ (found using Algorithm 1) with the deterministic solution. Specifically, we are interested in the quantity:

$$\frac{X_{\infty,\text{deterministic}} - E[Y_{\tau,\text{stochastic}}]}{P} \tag{18}$$

where τ is as defined in eqn(12), for large values of P. In the following plots, the parameter η (stochastic) is set to 1, as well as the parameters β_i (deterministic). Note that although we only have theoretical grounds for expecting the means to converge when the initial conditions are posed as a proportion of the population P, this behavior is also exhibited for fixed numbers as the population grows. We expect the convergence to be monotone in the cases where the problem is posed as proportions; Case II shows non-montonicity which may result if the initial conditions are posed as fixed numbers.

CASE I: S_0 is set at 1, and L_0 is set at 1

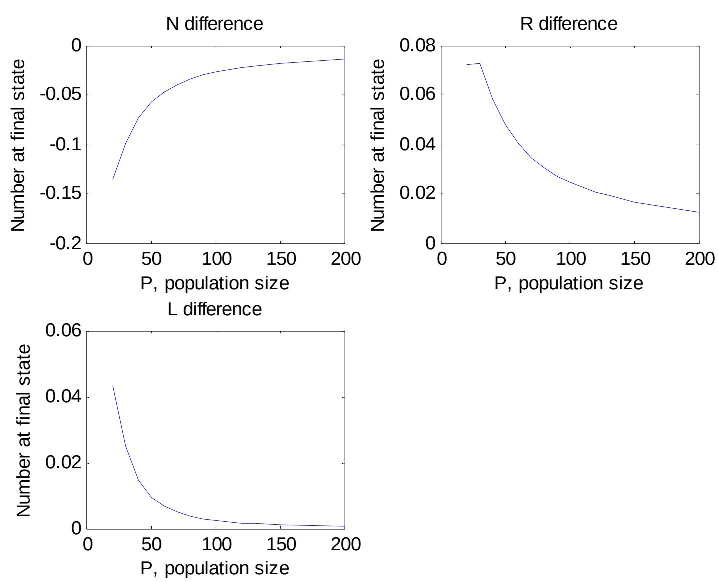

Figure 27. Case I convergence of Means

CASE II: S_0 is set to 1, and L_0 is set to 10% of the population

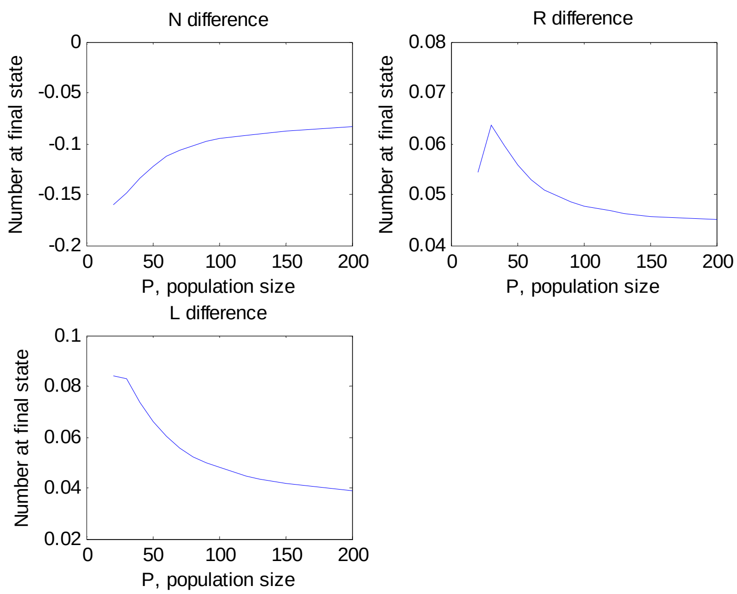

Figure 28. Case II convergence of Means

CASE III: S_0 is set to 10% and L_0 is set to 1

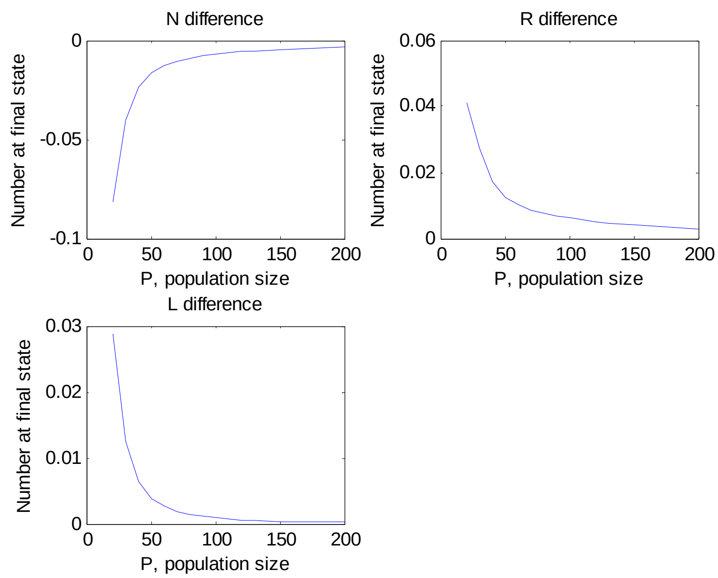

Figure 29. Case III convergence of Means

CASE IV: S_0 is set to 1 and L_0 is set to 30% of the population

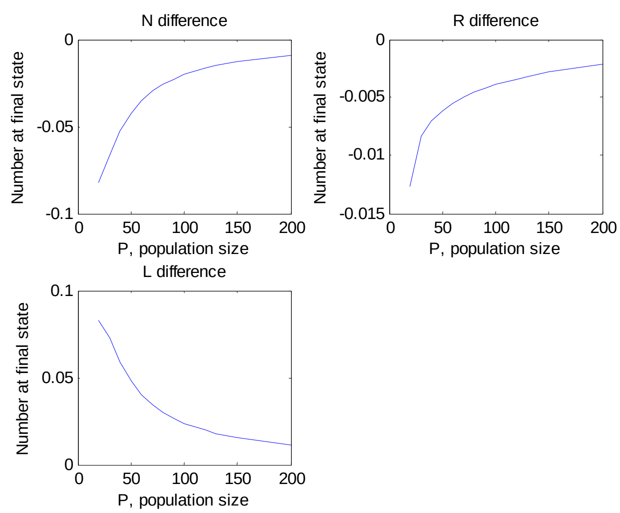

Figure 30. Case IV convergence of Means

In each case presented here, the quantity defined in equation (18) tends toward zero with increasing population size, as expected. We feel confident in stating that our numerical results for means does agree with Barbour (1972) and (1973) papers.

We now illustrate, through various choices of initial conditions and rate parameters, the fact that the distribution of the number of individuals in classes R and L

at the time the number of individuals in class S becomes 0 is well approximated by a multivariate normal distribution as the population size increases.

Now we consider a run of the model with the parameters P = 100, $\eta = 1$. The bivariate distribution of the number of individuals in classes R and L when the number of individuals in class S becomes 0 under different initial conditions are plotted here:

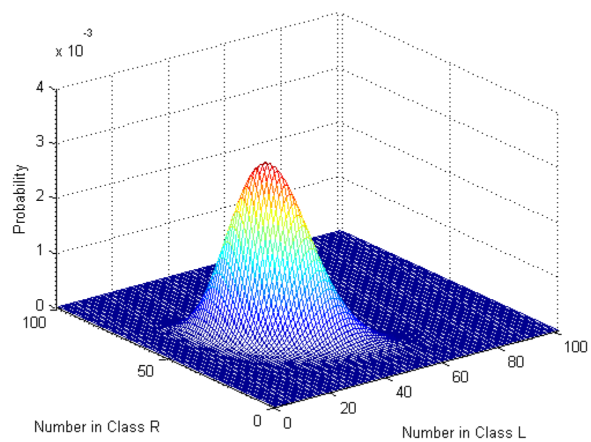

Figure 31. $P = 100, S_0 = 10, L_0 = 10, \eta = 1$

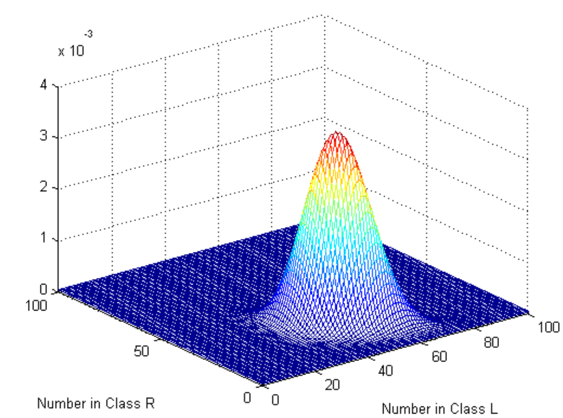

Figure 32. $P = 100, S_0 = 20, L_0 = 20, \eta = 1$

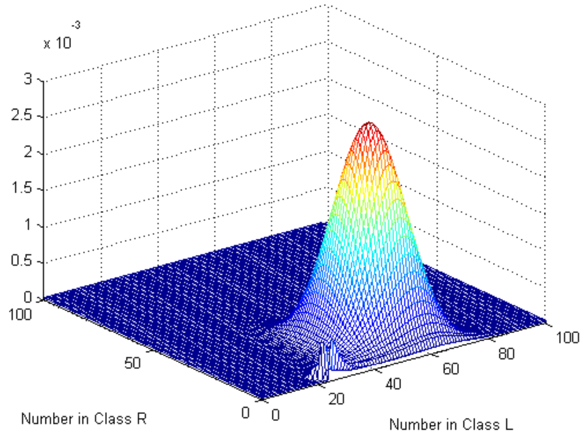

Figure 33. $P = 100, S_0 = 5, L_0 = 20, \eta = 1$

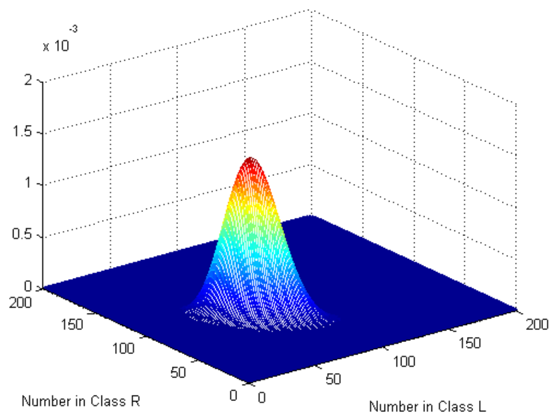

Figure 34. $P = 200, S_0 = 15, L_0 = 20, \eta = 1$

These plots illustrate that for some initial conditions as specified in Equation (13), the distribution of the number of individuals in each class R and L when the number of individuals in class S becomes 0 is approximately multivariate Normal. However, note the apparent multimodal behavior displayed in figure 33. We make this assertion based on the theoretic framework in Barbour (1972 and 1973). In essence, we are applying an extension of the Central Limit Theorem to study the expected value of a more general Markov process.

Barbour (1972) provides a procedure for computing the variance/covariance structure of the limiting Gaussian distribution of proportion of individuals in each absorbing class. An application of this method is beyond the scope of this thesis.

E. SUMMARY

Our results bring home that for the areas of most interest for persuading/changing public opinion, the deterministic model represents the mean outcome of the stochastic propagation process. In the situation where the ideological spreading does not die out in its initial stage, the results of this chapter indicate that the final proportion of people in each of the ideological classes is approximately Normally distributed with a mean given by the solution of a system of ODEs similar to those developed in Chapter III and a covariance structure that can be computed from the rate parameters and the initial conditions.

We believe that the major contribution of this chapter is to provide a framework to explore models of ideology propagation. The interaction rules chosen for this particular development are reasonable and fit in smoothly with other models discussed in the literature.

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V. CONCLUSION

A. RESULTS

In this thesis we have extended the models of rumor propagation to the case of a competitive 'hearts and minds' campaign. We have developed a general methodology and applied it to study two cases.

In Chapter III we develop two deterministic models, the ***NSRL*** and ***NSCRL***, which consider the cases of a subversive element and a political campaign, respectively. We explore the behavior under a wide range of choices for parameters and initial conditions.

In Chapter IV we study a stochastic model related to the ***NSRL*** model. We cite an important theoretical result which shows that the distribution of the number of individuals in each class of the ***NSRL*** model (and all models similarly posed) when the propagation of the ideology ceases can usefully be approximated by a multivariate Normal distribution with a mean coincident with that determined by the deterministic NSRL model. This distributional quality is conditional on fairly weak conditions on the initial states. This makes our approach a compliment, and in some cases an alternative, to agent-based simulation of this social phenomena.

The most important result from our work is not the ***NSRL*** and ***NSCRL*** models themselves, but the methodology that was applied in creating them. The framework developed here may be used to study a competitive ideological process with a large variety of dynamics. What we present here is a small piece of what is possible.

As shown in Chapter III, Figure 10, a *large* increase in initial supporters is required to overcome a *relatively modest* increase in contrarians. This shows that given the rules in the *NSRL* model, the supporter side is operating at a significant disadvantage. This is an important insight into *Sea Shaping* and other operations that the US is currently involved in.

Secondly, we show that for large populations, for a fixed initial number of supporters, parity is reached if less than 10% of the population is initially counter. In Chapter III, we define parity as the 'break even' point, where the number of pro-

individuals is equal to the number of counter- individuals at the end of propagation. Operationally, the counter- side will always need at least 10% of the population to prevent spread of the ideology for any number of supporters.

B. FUTURE RESEARCH

In our opinion, this Thesis contains many possibilities for both applied mathematicians and probability theorists. Many mathematical subtleties have been left unexplored. Additionally, in introducing a new class of models we have painted with a small brush and considered only a few special cases. Many extensions are possible. We feel that natural continuations of this work would include the following:

1. Application of Barbour's method for computing the variance of the absorbing states distribution.

2. We have only considered the *asymptotic* behaviors of the models proposed. We have not developed any Measures of Effectiveness (MOE's) which take the *time-dependant* behavior of the system into consideration. These new MOE's could be studied using the deterministic models proposed in Chapter III.

3. Throughout, we have assumed constant population sizes with homogeneous mixing. There is no reason that the theory developed here could not be extended to cases where the mixing parameters (β_i) are continuous functions in time, or to cases with changing population size due to individuals joining the population, leaving the population, or both.

4. We have not fully considered the impact of mass media. We propose that a mechanism which would deal with the effect of mass media would be to use an artificial redistribution of the population at some time in the model. This would be accomplished the same way that Lanchester's equations with reinforcements are modeled.

5. We were not able to find an adequate data set on which to fit parameters. If one could be found, a fitting to the data and hypothesis testing would be an important project.

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APPENDIX 1: IMPLEMENTATION OF ALGORITHIM I (MATLAB)

```
function []=Vectors(n,Io,Co, plot)
%% Code to compute the joint probability distribution of the absorbing
%% states in the NSRL model.  Developed by Roberto Szechtman with
Harrison
%% Schramm.

%% Uses equations (10, 11) from Chapter IV.

% Input variables:
% n:  population size (integer)
% Io: Initial pro-spreaders (integer)
% Co: Inital counter-spreaders (integer)
% plot:  Logical (real, treated as boolean). A value of zero suppresses
% plots, a value greater than zero produces plots.

% Output Variables
% left void in on this page, but any of the following could be placed
in
% the brackets immediately after 'function'

%mean_s, mean_r, mean_c:  The expected value of these at the end state,
%computed directly from the definition of expected value

%Var_S, Var_R, Var_C:  The variance of these at the end state, computed
%directly from the definition of varaince

% Joint:  The joint distribution of classes R and C at the absorbing
state.


s=n-Io-Co; i=Io; c=Co;
pr=zeros(n+1,n+1,n+1);
pr(s,i,c)=1; % Working probabilities
fin=zeros(n+1,n+1); %% end state of (S >0 and C>0) [s,c]
fin2=zeros(n+1,n+1); %%  (Temporary, S = 0) (not used as of now)

while(sum(sum(sum(pr)))>0) %% while any probability of forward
transition exists.
    tmp=zeros(n+1,n+1,n+1); % last state;
    [s,i,c]=ind2sub(size(pr),find(pr>0)); % provides col vectors of
S,i,c prob >0;
    r=n+1-s-i-c;
    total=(s>0).*(i>0).*s.*i+(i>1)*.5.*i.*(i-
1)+(r>0).*(i>0).*i.*r+(c>0).*(i>0).*i.*c;
    total=(total==0)+total;
    p1=(s>0).*(i>0).*s.*i./total; %vectors that contain the
probabilities moving forward for each element.
    p2=(i>1)*.5.*i.*(i-1)./total;
    p3=(r>0).*(i>0).*i.*r./total;
```

```
    p4=(c>0).*(i>0).*i.*c./total;

    s1=s(s>1); i1=i(s>1); c1=c(s>1); p11=p1(s>1);
    if(size(s1)>0)
    tmp(sub2ind(size(tmp),s1-1,i1+1,c1))=tmp(sub2ind(size(tmp),s1-
1,i1+1,c1))+p11.*pr(sub2ind(size(pr),s1,i1,c1));
    end
    s1=s(s==1); i1=i(s==1); c1=c(s==1); p11=p1(s==1);
    if(size(s1)>0)

fin2(sub2ind(size(fin2),i1+1,c1))=fin2(sub2ind(size(fin2),i1+1,c1))+p11
.*pr(sub2ind(size(pr),s1,i1,c1));
    end

    s2=s(i>2); i2=i(i>2); c2=c(i>2); p22=p2(i>2);
    if(size(s2)>0)
        tmp(sub2ind(size(tmp),s2,i2-2,c2))=tmp(sub2ind(size(tmp),s2,i2-
2,c2))+p22.*pr(sub2ind(size(pr),s2,i2,c2));
    end
    s2=s(i==2); i2=i(i==2); c2=c(i==2); p22=p2(i==2);
    if(size(s2)>0)

fin(sub2ind(size(fin),s2,c2))=fin(sub2ind(size(fin),s2,c2))+p22.*pr(sub
2ind(size(pr),s2,i2,c2));
    end

    s3=s(i>1); i3=i(i>1); c3=c(i>1); p33=p3(i>1);
    if(size(s3)>0)
        tmp(sub2ind(size(tmp),s3,i3-1,c3))=tmp(sub2ind(size(tmp),s3,i3-
1,c3))+p33.*pr(sub2ind(size(pr),s3,i3,c3));
    end
    s3=s(i==1); i3=i(i==1); c3=c(i==1); p33=p3(i==1);
    if(size(s3)>0)

fin(sub2ind(size(fin),s3,c3))=fin(sub2ind(size(fin),s3,c3))+p33.*pr(sub
2ind(size(pr),s3,i3,c3));
    end

    s4=s(i>1); i4=i(i>1); c4=c(i>1);p44=p4(i>1);
    if(size(s4)>0)
        tmp(sub2ind(size(tmp),s4,i4-
1,c4+1))=tmp(sub2ind(size(tmp),s4,i4-
1,c4+1))+p44.*pr(sub2ind(size(pr),s4,i4,c4));
    end

    s4=s(i==1); i4=i(i==1); c4=c(i==1); p44=p4(i==1);
    if(size(s4)>0)

fin(sub2ind(size(fin),s4,c4+1))=fin(sub2ind(size(fin),s4,c4+1))+p44.*pr
(sub2ind(size(pr),s4,i4,c4));
    end

    pr=tmp;
end
```

```
%now take of the mass associated with s=0 and i>0
clear pr;
fin3=zeros(n+1,1);
pr=fin2; %% fin2 holds the case where s = 0, i > 0, C>0.
while(sum(sum(pr))>0)
    tmp=zeros(n+1,n+1);
    [i,c]=ind2sub(size(pr),find(pr>0));
    r=n+1-i-c;
    total=(i>1)*.5.*i.*(i-1)+(r>0).*(i>0).*i.*r+(c>0).*(i>0).*i.*c;
    total=(total==0)+total;

    p2=(i>1)*.5.*i.*(i-1)./total;
    p3=(r>0).*(i>0).*i.*r./total;
    p4=(c>0).*(i>0).*i.*c./total;



    i2=i(i>2); c2=c(i>2); p22=p2(i>2);
    if(size(i2)>0)
        tmp(sub2ind(size(tmp),i2-2,c2))=tmp(sub2ind(size(tmp),i2-
2,c2))+p22.*pr(sub2ind(size(pr),i2,c2));
    end
    i2=i(i==2); c2=c(i==2); p22=p2(i==2);
    if(size(i2)>0)

fin3(sub2ind(size(fin3),c2))=fin3(sub2ind(size(fin3),c2))+p22.*pr(sub2i
nd(size(pr),i2,c2));
    end

    i3=i(i>1); c3=c(i>1); p33=p3(i>1);
    if(size(i3)>0)
        tmp(sub2ind(size(tmp),i3-1,c3))=tmp(sub2ind(size(tmp),i3-
1,c3))+p33.*pr(sub2ind(size(pr),i3,c3));
    end
    i3=i(i==1); c3=c(i==1); p33=p3(i==1);
    if(size(i3)>0)

fin3(sub2ind(size(fin3),c3))=fin3(sub2ind(size(fin3),c3))+p33.*pr(sub2i
nd(size(pr),i3,c3));
    end

    i4=i(i>1); c4=c(i>1);p44=p4(i>1);
    if(size(i4)>0)
        tmp(sub2ind(size(tmp),i4-1,c4+1))=tmp(sub2ind(size(tmp),i4-
1,c4+1))+p44.*pr(sub2ind(size(pr),i4,c4));
    end
    i4=i(i==1); c4=c(i==1); p44=p4(i==1);
    if(size(i4)>0)

fin3(sub2ind(size(fin3),c4+1))=fin3(sub2ind(size(fin3),c4+1))+p44.*pr(s
ub2ind(size(pr),i4,c4));
    end

    pr=tmp;
end
%% fin3 holds i = 0;
```

```
fin; fin3;
mean_s=sum((1:n+1)*fin);

Var_S = sum((1:n+1).^2 * fin) - mean_s^2;
mean_c=sum(fin*(1:n+1)')+(1:n+1)*fin3;
Var_C = sum(fin*((1:n+1).^2)') +(1:n+1).^2*fin3 - mean_c^2;
mean_r= n - mean_s - mean_c;
sum(sum(fin));, sum(sum(fin3));

% fin holds s,c
%% fin3 holds c distribution.

joint = zeros(n+1,n+1); % r and c matrix

for c=1:n
   for r=1:n
      if(n+1-r-c>0)
         joint(r,c)=fin(n+1-r-c,c);%s>0, r>0, c>0
      end
   end
end

for c=1:n
   joint0(c)=fin(n+1-c,c); %s>0, r=0, c>0
    joint(n+1-c,c)=joint(n+1-c,c)+fin3(c);%s=0, r>0, c>0
end
joint00=fin3(n+1);%s=0, r=0, c=n+1

sum(sum(joint))+sum(joint0)+joint00;

j = sum(joint');
N2 = ((1:n+1).^2)';

Var_R = (j*N2) - mean_r^2;

if(plot >0)

handle1= figure;
mesh(joint);
xlim([0 n+1]);
ylim([0 n+1]);
xlabel('Number in class C');
ylabel('Number in class R');
zlabel('Probability');
end
```

APPENDIX II: MATLAB CODE TO PERFORM BINARY SEARCH FOR CRITICAL PROPORTION OF CONTRARIANS

```
function[num rout] = estimate(r0,tol, minN, maxN, Isize, step)

% July 2006 Harrison Schramm
% works with SIRC2.m written by Harrison Schramm and Carlos Borges

% Determines the number of initial contrarians (ro) to achieve parity
% for a wide range of populations and a fixed Io

% Input Parameters
%  r0:  Initial guess for the proportion
%  tol: Numerical tolerance for binary search
%  minN, maxN:  Bounds on the size of the population
%  Isize:  Io (fixed)
%  step:  Number of samples to compute on the interval.

%  Returns rout, which is a vector of the values of rc found for the
%  different population sizes and
%  num which is the vector of population sizes

num = linspace(minN, maxN, step);
rout = ones(1,length(num));
maxruns = 100;  % prevents infinite loops
for i = 1:length(num)
   rlow = max(0,r0-.05);
   rhi = min(1,r0 + .05);
   rout(i) = r0;
   ratio = 2;
   goagain = true;
   count = 0;


   while(goagain)
     rout(i) = (rhi + rlow)/2;
     z = num(i);
     y = rout(i);
     m = SIRC2(y,z, Isize);
     l = length(m);
     count = count +1;
     ratio = m(l,4)/m(l,3);
        if(count >maxruns)
            goagain = false

        end
        if (abs(ratio - 1) < tol)
                goagain = false;

        end
        if(ratio <1)
             rlow = rout(i);
        end
        if(ratio > 1)
```

```
              rhi = rout(i);
        end


    end
end


n = num;
n;
rout;

handle1 = figure()

plot(n,rout)
title('Critical proportion of contrarians vs. population size')
xlabel('population size')
ylabel('ro, proportion of contrarians')


%______________

function [y] = SIRC2(r,n,io)
%
%2006 Harrison Schramm and Carlos Borges

% y(1) = S
% y(2) = I
% y(3) = R
% y(4) = C

%Initial conditions
IC = [((1-r)*n -io);io;0;(r*n)];

[t y] = ode45(@f,[0;100],IC);

%-------------------------------------
function dydt = f(t,y)

 dydt = [-y(1)*y(2)
         y(2)*(y(1)-y(2)-y(3)-y(4))
         y(2)*(y(2)+y(3))
         y(2)*y(4)];
```

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